Lösungsvorschläge für 9 Übungsblatt (1.)

9.1. • $\int_0^{\pi} x \sin x \, dx = \begin{bmatrix} f(x)=x & , \; f'(x)=1 \\ g'(x)=\sin x & , \; g(x)=-\cos x \end{bmatrix}$

$= x\cdot(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} 1\cdot(-\cos x)\,dx$

partielle Integration

$= -x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx$

$= -x\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi} = -\pi\cos\pi - (-0\cdot\cos 0) + \sin\pi - \sin 0$

$= -\pi\cdot(-1) = \pi = \underline{3,1415...}$

• $n=4, \quad h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

$x_0 = 0, \; x_1 = x_0 + h = 0 + \frac{\pi}{4} = \frac{\pi}{4}, \; x_2 = x_0 + 2h = 0 + \frac{2\cdot\pi}{4} = \frac{\pi}{2}, \; x_3 = x_0 + 3h = \frac{3\pi}{4}, \; x_4 = \pi.$

$Qf_b = h\sum_{j=1}^{n} f(x_j) = \frac{\pi}{4}\left(\frac{\pi}{4}\cdot\sin\frac{\pi}{4} + \frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3\pi}{4}\sin\frac{3\pi}{4} + \pi\sin\pi\right) = \underline{2,9784...}$

Rechteckregel

9.2. $f(x) = \frac{10}{5x-1}$

• $I = \int_2^4 \frac{10}{5x-1}\,dx = \begin{bmatrix} t := 5x-1 \\ dt = 5\,dx \\ t(2) = 5\cdot 2 - 1 = 9 \\ t(4) = 5\cdot 4 - 1 = 19 \end{bmatrix} = 2\int_2^4 \frac{5\,dx}{5x-1} = 2\int_9^{19} \frac{dt}{t} = 2\left(\log 19 - \log 9\right) = \underline{1,49442...}$

• $n=4, \quad h = \frac{b-a}{n} = \frac{4-2}{4} = 0.5$

$x_0 = 2, \; x_1 = 2.5, \; x_2 = 3, \; x_3 = 3.5, \; x_4 = 4; \quad y_0 = \frac{10}{5\cdot 2-1} = \frac{10}{9}, \; y_1 = \frac{10}{5\cdot 2.5 - 1} = \frac{20}{23}, \; y_2 = \frac{5}{7},$

$y_3 = \frac{20}{33}, \; y_4 = \frac{10}{19}$

$T_n = h\cdot\left(\frac{1}{2}y_0 + \sum_{j=1}^{n-1} y_j + \frac{1}{2}y_n\right)$

$T_4 = h\cdot\left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right) = 0.5\cdot\left(\frac{1}{2}\cdot\frac{10}{9} + \frac{20}{23} + \frac{5}{7} + \frac{20}{33} + \frac{1}{2}\cdot\frac{10}{19}\right)$

$= \frac{911135}{605682} = \underline{1,50431...}$

2.

- $$\left|T_n - I\right| \leq \frac{h^2}{12} \max_{x\in[a,b]} |f''(x)|(b-a)$$

$$f'(x) = \left(\frac{10}{5x-1}\right)' = 10\cdot\left(\frac{1}{5x-1}\right)' = 10\cdot\frac{-1}{(5x-1)^2}\cdot(5x-1)' = \frac{-50}{(5x-1)^2}$$

$$f''(x) = \left(\frac{-50}{(5x-1)^2}\right)' = -50\cdot\frac{-1}{(5x-1)^4}\cdot 2(5x-1)\cdot 5 = \frac{500}{(5x-1)^3}$$

$$|T_4 - I| \leq \frac{1}{4\cdot 12} \max_{x\in[2,4]} \left|\frac{500}{(5x-1)^3}\right| (4-2) = \frac{1}{24} \max_{x\in[2,4]} \frac{500}{(5x-1)^3} =$$

$$= \frac{1}{24}\cdot\frac{500}{(5\cdot 2-1)^3} = \frac{500}{24\cdot 9^3} = \frac{125}{4374} = 0.02857\ldots$$

$$h = \frac{b-a}{n}$$

- $$\frac{(b-a)^2}{n^2}\cdot \max_{x\in[a,b]} |f''(x)|(b-a) \leq 10^{-4}$$

$$\frac{(b-a)^3}{n^2}\cdot \max_{x\in[a,b]} |f''(x)| \leq 10^{-4}$$

$$\frac{(4-2)^3}{n^2}\cdot\frac{500}{729} \leq 10^{-4}$$

$$n^2 \geq \frac{8\cdot 500}{729\cdot 10^{-4}} = \frac{40\cdot 10^6}{729} = 54869.7\ldots$$

$$n \geq 234.243\ldots \quad \text{oder} \quad n \leq -234.243\ldots$$

Das kleinste $n \in \mathbb{N}$, so dass $|T_4 - I| \leq 10^{-4}$ gilt, ist $n = 235$.

9.1 b), 9.2. b), 9.3 b) analog!

9.3.

- $S_4 = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$

$= \frac{1}{2\cdot 3}\left(\frac{10}{9} + 4\cdot\frac{20}{23} + 2\cdot\frac{5}{7} + 4\cdot\frac{20}{33} + \frac{10}{19}\right)$

aus 9.2a).

$= \frac{1358015}{908523} = 1.4947\ldots$

- $\left|S_{2k} - I\right| \leq \frac{h^4}{180} \max_{x\in[a,b]} \left|f^{(4)}(x)\right| (b-a)$

$\left|S_4 - I\right| \leq \frac{\left(\frac{1}{2}\right)^4}{180} \cdot (4-2) \cdot \max_{x\in[2,4]} \left|\left(\frac{10}{5x-1}\right)^{(4)}_x\right|$

$= \frac{1}{90\cdot 16} \max_{x\in[2,4]} \frac{150\,000}{(5x-1)^5} = \frac{150000}{90\cdot 16\cdot 9^5} = \frac{625}{354294} = 0{,}01764\ldots$

$\left(\frac{10}{5x-1}\right)''' = \left(\frac{500}{(5x-1)^3}\right)' = \frac{-500}{(5x-1)^{\cancel{6}\,4}} \cdot 3\cancel{(5x-1)}^2 \cdot 5 = \frac{-7500}{(5x-1)^4}$

$\left(\frac{10}{5x-1}\right)^{(4)} = \left(-\frac{7500}{(5x-1)^4}\right)' = \frac{7500}{(5x-1)^{\cancel{8}\,5}} \cdot 4\cdot\cancel{(5x-1)}^3 \cdot 5 = \frac{150000}{(5x-1)^5}$

- $\frac{(b-a)^5}{n^4\, 180} \cdot \max_{x\in[2,4]} \left|\left(\frac{10}{5x-1}\right)^{(4)}_x\right| \leq 10^{-4} \qquad n = ?$

$h = \frac{b-a}{n}$

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$n^4 \geq \frac{2^5 \cdot 150000}{180\cdot 10^{-4}\cdot 9^5} = \frac{800\,000\,000}{177147} = 4516{,}02\ldots$

$n \geq 8{,}19764 \qquad -\!\!\parallel\!\!- \quad n = 10$ (n muss gerade sein)