Lösungsvorschläge für 8. Übungsblatt

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8.1. In unserem Beispiel $f(x)=x^2$ wählen wir eine äquidistante Zerlegung:

$Z_n: a=x_0<x_1<\dots<x_n=b$

mit $x_j=a+\frac{j}{n}(b-a)$ für $0\le j\le n$ und den Stützstellen $\xi_j=x_j$. Dann folgt

$$S(Z_n,f,\xi)=\sum_{j=1}^{n} x_j^2(x_j-x_{j-1})$$

$$=\frac{b-a}{n}\left(na^2+\frac{2a}{n}(b-a)\sum_{j=1}^{n} j+\frac{(b-a)^2}{n^2}\sum_{j=1}^{n} j^2\right)$$

$$=a^2(b-a)+\frac{\cancel{2}a(b-a)^2}{n^2}\cdot\frac{(n+1)n}{\cancel{2}}$$

$$+\frac{(b-a)^3 n(n+1)(2n+1)}{6n^3}\xrightarrow[n\to\infty]{}\frac{b^3-a^3}{3}.$$

Also folgt $\int_a^b x^2dx=\lim_{n\to\infty}\left(\frac{b^3-a^3}{3}+O\left(\frac{1}{n}\right)\right)=\frac{b^3-a^3}{3}$.

Hier wir haben benutzt, dass $\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$ und $\sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6}$.

(Was lässt sich leicht mit der Induktion zeigen).

8.2. a) $\int c\sin x+2e^x-\frac{1}{x}=\int c\sin x\,dx+\int 2e^x dx+\int\left(-\frac{1}{x}\right)dx$

$$=c\int\sin x\,dx+2\int e^x dx-\int\frac{1}{x}dx$$

$$=-a\cos x+2e^x-\ln|x|$$

b) $\int\sqrt{3x^2-1}\cdot x\,dx=\left[3x^2-1=u,\ du=6x\,dx,\ dx=\frac{du}{6x}\right]$

$$=\int\sqrt{u}\cdot x\frac{1}{6x}du=\int\sqrt{u}\cdot\frac{1}{6}\cdot du$$

$$=\frac{1}{6}\int\sqrt{u}\,du=\frac{1}{6}\int u^{\frac{1}{2}}du=\frac{1}{6}\cdot\frac{2}{3}u^{\frac{3}{2}}=\frac{1}{9}u^{\frac{3}{2}}$$

$$=\frac{1}{9}(3x^2-1)^{\frac{3}{2}}$$

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c) $\int_{-2}^{4} |x| x \, dx = \int_{-2}^{0} |x| x \, dx + \int_{0}^{4} |x| x \, dx$

$= \int_{-2}^{0} (-x^2) \, dx + \int_{0}^{4} x^2 \, dx$

$= -\int_{-2}^{0} x^2 dx + \int_{0}^{4} x^2 dx = -\frac{x^3}{3}\Big|_{-2}^{0} + \frac{x^3}{3}\Big|_{0}^{4}$

$= -\frac{0^3}{3} - \left(-\frac{(-2)^3}{3}\right) + \left(\frac{4^3}{3} - \frac{0^3}{3}\right) = -\frac{8}{3} + \frac{64}{3} = \frac{56}{3}$

d) $\int_{0}^{\frac{\pi}{2}} \frac{\cos^3 x}{1-\sin x} dx = \left[\begin{array}{l} t = \sin x, \ dt = \cos x \, dx \\ \sin(0) = 0, \ \sin(\frac{\pi}{2}) = 1 \end{array}\right]$

$= \int_{0}^{\frac{\pi}{2}} \frac{\cos^2 x}{1-\sin x} \cos x \, dx = \int_{0}^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1-\sin x} \cos x \, dx$

$= \int_{0}^{1} \frac{1-t^2}{1-t} dt = \int_{0}^{1} (1+t) dt = \left(t + \frac{t^2}{2}\right)\Big|_{0}^{1}$

$= \left(1 + \frac{1^2}{2}\right) - \left(0 + \frac{0^2}{2}\right) = \frac{3}{2}$

e) $\int \frac{1}{x^2-2x} dx =$

$\frac{1}{x^2-2x} = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2) + Bx}{x(x-2)} = \frac{(A+B)x - 2A}{x(x-2)}$

Partialbruchzerlegung

$\begin{cases} A+B = 0 \\ -2A = 1 \end{cases} \iff \begin{cases} A = -\frac{1}{2} \\ B = -A = \frac{1}{2} \end{cases}$.

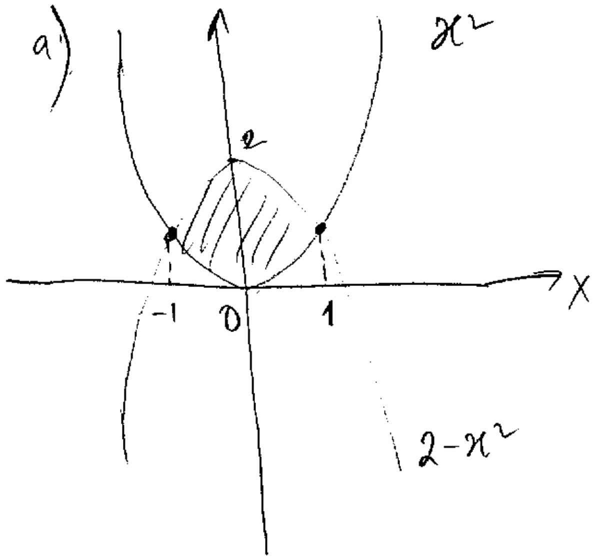
$= \int \frac{-\frac{1}{2}}{x} dx + \int \frac{\frac{1}{2}}{x-2} dx = -\frac{1}{2}\int \frac{1}{x} dx + \frac{1}{2}\int \frac{1}{x-2} dx = -\frac{1}{2}\log|x| + \frac{1}{2}\log|x-2|$

f) $\int_{0}^{1} x \exp(x) \, dx = \left[\begin{array}{l} \int_0^1 u \, dv = uv\Big|_0^1 - \int_0^1 v \, du \\ u = x, \ dV = \exp x \, dx, \ du = dx, \ v = \exp(x) \end{array}\right]$ Partielle Integration!

$= x \cdot \exp(x)\Big|_0^1 - \int_0^1 \exp(x) \, dx = x \exp(x)\Big|_0^1 - \exp(x)\Big|_0^1 =$

$= (1 \cdot e^1 - 0 \cdot e^0) - (e^1 - e^0) = 1$

3. a)



Integrationsgrenzen:

$x^2 = 2 - x^2$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

$$S = \int_{-1}^{1}\left(\left(2-x^2\right)-x^2\right)dx = 2\int_{-1}^{1}\left(1-x^2\right) = 2\cdot\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 2\cdot\left(\left(1-\frac{1}{3}\right)-\left(-1-\frac{(-1)^3}{3}\right)\right)$$

$$= 2\cdot\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$$

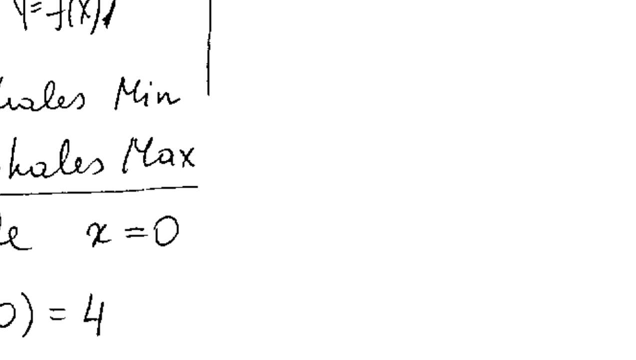
$$= 2\cdot\left(2-\frac{2}{3}\right) = 2\cdot\frac{4}{3} = \frac{8}{3}$$

b) Kritische Punkte:

$f'(x) = \left(x^3-3x^2+4\right)' = 3x^2-6x = 0$

$3x(x-2) = 0$

$x=0 \vee x=2$



$f''(x) = 6x-6$

$f''(2) = 6\cdot 2-6 > 0 \Rightarrow x=2$ lokales Min

$f''(0) = 6\cdot 0-6 < 0 \Rightarrow x=0$ lokales Max

Die Gleichung der Tangente an der Stelle $x=0$

$t(x) = f(0)+f'(0)\cdot(x-0) = 4+0\cdot(x-0) = 4$

Der gesuchte Flächeninhalt:

$$S = \int_0^3\left(4-\left(x^3-3x^2+4\right)\right)dx = -\int_0^3\left(x^3-3x^2\right)dx = -\left(\frac{x^4}{4}-x^3\right)\Big|_0^3 = -\left(\frac{3^4}{4}-3^3-\underbrace{\left(\frac{0^4}{4}-0^3\right)}_{=0}\right) = \frac{27}{4}$$

8.4.

a) $\int \frac{2x-3}{x^2-3x+1}\,dx = \begin{bmatrix} u := x^2-3x+1 \\ du = (x^2-3x+1)'dx = (2x-3)dx \end{bmatrix}$

$= \int \frac{du}{u} = \log|u| = \log|x^2-3x+1|.$

b) $\int \frac{x}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} u := 1-x^2 \\ du = -2x\,dx, \quad dx = \frac{du}{-2x} \end{bmatrix}$

$= \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{1}{-2\cancel{x}}\,du = -\frac{1}{2} \int \frac{du}{\sqrt{u}}$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}u^{-\frac{1}{2}+1} \cdot \left(-\frac{1}{2}+1\right)^{-1} = -\frac{1}{2} \cdot u^{\frac{1}{2}} \left(\frac{1}{2}\right)^{-1}$

$= -u^{\frac{1}{2}} = -\sqrt{1-x^2}.$

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```
#Monte Carlo Methode
a = 0.                        # Untere Grenze
b = 1                         # Obere Grenze
versuche = 100                # Anzahl der Versuche
f(x) = x^2                    # Integrand
gutewurfe = 0.
wurfe=0.
for i in range (1, versuche):
    x = random()
    y = random()
    if y <= f(x) :
        gutewurfe = gutewurfe + 1
    wurfe = wurfe + 1
integral = gutewurfe/wurfe
print integral
```

0.313131313131313

```
N(integral(x^2, x, 0, 1))
```

evaluate

0.333333333333333