International Summer School: An Introduction to Functional Analysis through Applications

FUNCTIONAL ANALYSIS AND CONTROL THEORY

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Dual spaces of the classical Banach spaces (1)

Given a normed space V, we denote by $V^\star$ its dual space. Recall that $V^\star$ is the space of all continuous linear mappings $f : V \to \mathbb{K}$. Let us look at some examples!

- We have $(\mathbb{K}^n)^\star = \mathbb{K}^n$ in the sense that each (continuous) linear mapping $\mathbb{K}^n \to \mathbb{K}$ is of the form $x \mapsto a_1x_1 + \cdots + a_nx_n$ with $a \in \mathbb{K}^n$.
- If V is a Hilbert space then $V^\star = V$ in the sense that each continuous linear mapping $V \to \mathbb{K}$ is of the form $x \mapsto \langle x, v\rangle$ with a fixed vector $v \in V$.
- As a special cases, each continuous linear mapping $f : \ell^2 \to \mathbb{K}$ is of the form $x \mapsto a_1x_1 + a_2x_2 + a_3x_3 + \cdots$ with $a \in \ell^2$.
- Let $p \geq 1$. Let $q := p/(p-1)$ so that $(1/p) + (1/q) = 1$. (If $p = 1$ then $q = \infty$.) Then $(\ell^p)^\star = \ell^q$ in the sense that each continuous linear mapping $\ell^p \to \mathbb{K}$ is of the form $x \mapsto a_1x_1 + a_2x_2 + a_3x_3 + \cdots$ with $a \in \ell^q$.

Dual spaces of the classical Banach spaces (2)

- To determine $(\ell^\infty)^\star$ is more complicated and requires some measure theory.
- Let $p \geq 1$. Let $q := p/(p-1)$ so that $(1/p) + (1/q) = 1$. (If $p = 1$ then $q = \infty$.) Then $\big(L^p(I)\big)^\star = L^q(I)$ in the sense that each continuous linear mapping $L^p(I) \to \mathbb{K}$ is of the form $f \mapsto \int_I f(x)g(x)\,\mathrm{d}x$ with $g \in L^q(I)$.
- The dual of $C(I)$ is the space of all regular Borel measures on I in the sense that each continuous linear functional $C(I) \to \mathbb{K}$ is of the form $f \mapsto \int_I f\,\mathrm{d}\mu$ for such a measure μ.

Exercise. Let c_0 be the space of all sequences in $\mathbb{K}$ which converge to zero. Show that c_0 is a closed subspace of ℓ^∞ (and hence a Banach space) and show that $(c_0)' = \ell^1$ in the sense that each continuous linear mapping $c_0 \to \mathbb{K}$ is of the form $x \mapsto a_1x_1 + a_2x_2 + a_3x_3 + \cdots$ with $a \in \ell^1$.

Extensions of linear functionals

Let V be a normed space and let U be a subspace of V. For each element $F \in V^\star$, the restriction $f = F|_U$ is an element of $U^\star$ with

$$\begin{aligned} \|f\| &= \sup\{F(u) \mid u \in U, \|u\| \leq 1\} \\ &\leq \sup\{F(v) \mid v \in V, \|v\| \leq 1\} = \|F\|. \end{aligned}$$

Conversely, we can ask whether or not a given element $f \in U^\star$ can be extended to an element $F \in V^\star$. That this is so is the contents of the famous Hahn-Banach theorem.

Hahn-Banach Extension Theorem. Given a normed space V, a subspace U and a continuous linear functional $f : U \to \mathbb{K}$, there is a continuous linear functional $F : V \to \mathbb{K}$ such that $F|_U = f$ and $\|F\| = \|f\|$.

This theorem was originally proved (in 1927) only for $\mathbb{K} = \mathbb{R}$. There are many generalizations, variations and refinements of this theorem.

Hans Hahn
(1879-1934)

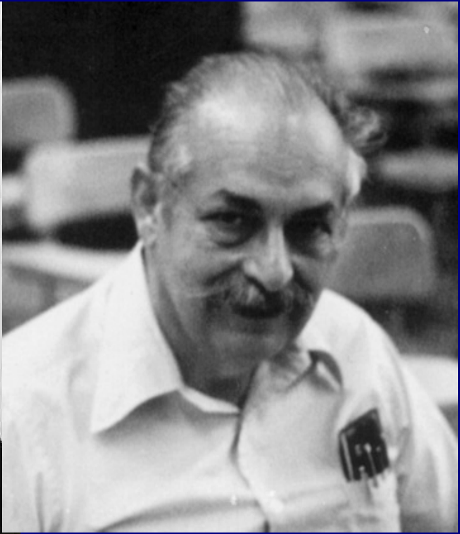
Stefan Banach
(1892-1945)

Weak topologies

The unit sphere $\{x \in V \mid \|x\| \leq 1\}$ of a normed space V is bounded and closed. If V is finite-dimensional this is equivalent to compactness. If V is infinite-dimensional, this is no longer true. On the other hand, compactness is a very desirable property; so one may try to weaken the topology to make the unit sphere compact. The **weak topology** on V is defined by letting $v_i \to v$ if and only if $f(v_i) \to f(v)$ for all $f \in V^\star$. On the dual space $V^\star$ an even weaker topology can be defined, the so-called weak* topology. In this topology we have $f_i \to f$ if and only if $f_i(v) \to f(v)$ for all $v \in V$ (pointwise convergence).

Banach-Alaoglu Theorem (1940) Let V be a Banach space. Then the unit sphere in $V^\star$ is weak*-compact.

Stefan Banach
(1892-1945)



Leonidas Alaoglu
(1914-1981)

Convexity

A subset C of a real vector space V is called **convex** if it contains, with any two points p and q, also the line segment

$$\{(1-t)p + tq \mid 0 \leq t \leq 1\}.$$

Intuitively, this means that C has no holes or indentations.

- The intersection of any family of convex sets is convex again.
- Any affine image of a convex set is convex again.
- If V is a topological vector space and if $C \subseteq V$ is convex, then the closure $\overline{C}$ is convex again.

Definition. Let X be an arbitrary subset of a real vector space V. The **convex hull** of X, denoted by $\mathrm{conv}(X)$, is the unique smallest convex set containing X (namely, the intersection of all convex sets containing X).

Control systems

Controlled dynamical system: $\dot{x}(t) = f\big(x(t), t, u(t)\big)$

- $x(t) =$ system state at time t
- $u(t) =$ value of the external control variable at time t
- $f =$ function which describes the time evolution of the system
- If f does not explicitly depend on time, the system is called **autonomous**.
- Once the function $t \mapsto u(t)$ is chosen, we simply have a nonautonomous system

$$\dot{x}(t) \;=\; F\big(x(t), t\big) \quad \text{where} \quad F(x,t) := f\big(x, t, u(t)\big)\,.$$

However, choosing the control u appropriately is the central task in control theory.

Example 1: Use of an insecticide

Control system:

$$\dot{x}(t) = k \cdot x(t) - u(t)$$

where

$$\begin{aligned} x(t) &= \text{size of an insect population at time } t \\ u(t) &= \text{application rate of an insecticide} \\ k &= \text{natural growth rate of the insect population} \\ &\quad\ \text{(assumed as known)} \end{aligned}$$

Control problem: Choose the function u to "influence" or "control" the insect population in a desired way (for example, to extinguish the population before the apple bloom starts).

Example 2: Rocket car

Control system:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + u(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where

$$\begin{aligned} x(t) &= \text{position of the car at time } t \\ y(t) &= \dot{x}(t) = \text{speed at time } t \\ u(t) &= m\ddot{x}(t) \quad (m = 1) \end{aligned}$$

Control problem: Drive the rocket car subject to a constraint $|u(t)| \leq u_{\max}$ on the possible acceleration (for example, to reach a prescribed position and velocity from a given initial position and velocity).

Example 3: Investment policy

Control system:

$$\dot{x}(t) = u(t)x(t)$$

where

$$x(t) = \text{production rate of a commodity (such as steel) at time } t$$

$$u(t) = \text{percentage to be re-invested into the production process at time } t$$

Control problem: Choose the function u, i.e., decide how much of the production should be re-invested rather than sold to reach certain business goals (for example, to generate a desired profit over a given planning interval $[0, T]$).

Example 4: Cancer Treatment (1)

Divide the cancer cells in an organ into three compartments according to their cell-cycle phase:

- first growth phase/dormant phase (prior to DNA reduplication);
- phase of DNA reduplication;
- second growth phase leading up to mitosis (cell division).

Let $N_i(t)$ be the number of cells in compartment i at time t and let $a, b, c > 0$ be the transition rates between the different compartments. Then

$$\begin{aligned}
\dot{N}_1(t) &= -a\,N_1(t) + 2c\,N_3(t) \\
\dot{N}_2(t) &= a\,N_1(t) - b\,N_2(t) \\
\dot{N}_3(t) &= b\,N_2(t) - c\,N_3(t)
\end{aligned}$$

if the cells are left to themselves.

Example 4: Cancer Treatment (2)

Assume that two types of medicine are used to fight the cancer:

- a killing agent which acts mainly on cells in the third compartment, which are particularly vulnerable;
- a blocking agent which acts on cells in the second compartment by blocking the enzyme which stimulates DNA reduplication.

Let $u(t)$ and $v(t)$ be the rates at which the killing agent and the blocking agent are administered, respectively. Simplifying, we have

$$\begin{aligned}
\dot{N}_1(t) &= -a\,N_1(t) + 2c\,N_3(t)\big(1-u(t)\big) \\
\dot{N}_2(t) &= a\,N_1(t) - b\,N_2(t)\big(1-v(t)\big) \\
\dot{N}_3(t) &= b\,N_2(t)\big(1-v(t)\big) - c\,N_3(t)
\end{aligned}$$

$$\begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \end{bmatrix} = \left(\begin{bmatrix} -a & 0 & 2c \\ a & -b & 0 \\ 0 & b & -c \end{bmatrix} + u \begin{bmatrix} 0 & 0 & -2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & -b & 0 \end{bmatrix} \right) \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

Reachability sets and their geometry

Control system:

$$\dot{x}(t) = f\big(x(t), t, u(t)\big) \qquad x(t_0) = x_0$$

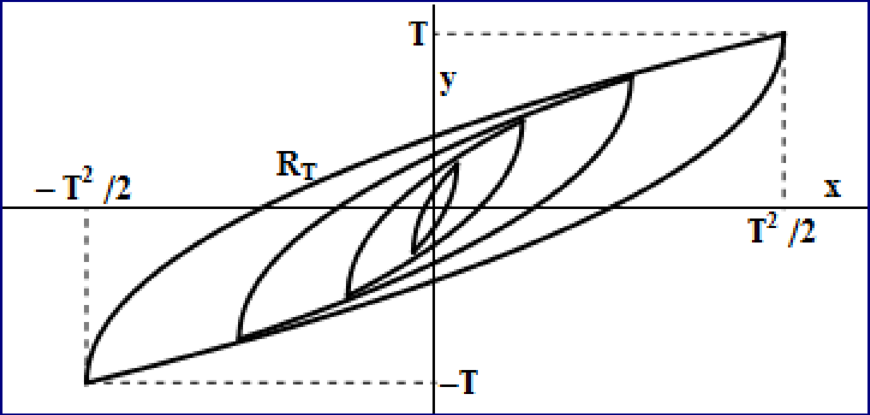
Reachability set at time $T > 0$:

$$R_T \;:=\; \{x_u(T) \mid u \text{ is an admissible control on } [0, T]\}$$

In words: R_T is the set of all states which can be reached starting from the initial state x_0 using an admissible control over the time interval $[0, T]$. It helps to visualize the boundary $\partial R(t)$ as a "wave front" evolving in time.

Example: rocket car problem

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad |u(t)| \leq 1$$



Reachability sets of linear control systems

Linear control system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \quad u_{\min} \leq u_i(t) \leq u_{\max}$$

Explicit solution formula:

$$x_u(t) \;=\; \Phi(t, t_0)x_0 + \Phi(t, t_0) \int_{t_0}^{t} \Phi(\tau, t_0)^{-1} B(\tau)u(\tau)\, \mathrm{d}\tau$$

- **Convexity:** The set of all admissible controls is convex, and the mapping $u \mapsto x_u(T)$ is affine. Hence R_T is convex (as an affine image of a convex set).
- **Compactness:** If we equip $L^\infty[0, T]$ with the $w^\star$-topology, the set of all admissible controls is compact (Banach-Alaoglu Theorem), and the mapping $u \mapsto x_u(T)$ is continuous. Hence R_T is compact (as a continuous image of a compact set).
- **Continuity:** The mapping $T \mapsto R_T$ is continuous with respect to the Hausdorff metric. Hence the reachability set varies continuously with time.

Solving the rocket car problem (1)

Fix $t > 0$. The solution formula yields

$$\begin{bmatrix} x_u(t) \\ y_u(t) \end{bmatrix} = \begin{bmatrix} x_0 + ty_0 + \int_0^t (t-s)u(s)\,\mathrm{d}s \\ y_0 + \int_0^t u(s)\,\mathrm{d}s \end{bmatrix}.$$

Thus the condition $\big(x_u(t), y_u(t)\big) = (0,0)$ reads

$$\begin{bmatrix} \int_0^t (t-s)u(s)\,\mathrm{d}s \\ \int_0^t u(s)\,\mathrm{d}s \end{bmatrix} = \begin{bmatrix} -x_0 - ty_0 \\ -y_0 \end{bmatrix}.$$

Define $T_u : L^1[0,t] \to \mathbb{R}$ by $T_u(f) := \int_0^t f(\tau)u(\tau)\,\mathrm{d}\tau$. Then $T_u(t - s) = -x_0 - ty_0$ and $T_u(1) = -y_0$, hence $T_u(s) = x_0$ and therefore

$$T_u(as + b) = ax_0 - by_0 =: \Lambda_t(as + b) \quad \text{for all } a, b \in \mathbb{R}.$$

Thus the condition that u steers the system to the target state in time t means that T_u coincides with Λ_t on the space P_t of all linear polynomial functions on the interval $[0, t]$.

Solving the rocket car problem (2)

Let $C_t := \|\Lambda_t\|_{op} = \| T_u |_{P_t} \|_{\mathrm{op}}$ so that

$$\begin{aligned} C_t &= \max\{|T_u(as+b)| \mid \|as+b\|_1 \leq 1\} \\ &= \max\{|ax_0 - by_0| \mid \int_0^t |as+b|\,\mathrm{d}s = 1\}. \end{aligned}$$

It is easily checked that $t \mapsto C_t$ decreases and tends to 0 as $t \to \infty$. (Interpretation: C_t is the minimal engine power required to make it possible to reach the target state in time t.) Assume the maximum is taken for $a = a^\star$ and $b = b^\star$. Then

$$\begin{aligned} C_t \;&=\; |a^\star x_0 - b^\star y_0| \;=\; |\Lambda_t(a^\star s + b^\star)| \\ &\leq\; \|\Lambda_t\|_{op} \cdot \|a^\star s + b^\star\|_1 \;=\; \|\Lambda_t\|_{op} \;=\; C_t. \end{aligned}$$

By the Hahn-Banach theorem, there is a norm-preserving extension of Λ_t from P_t to $L^1[0,t]$. This extension is necessarily of the form T_u for some $u \in L^\infty[0,t]$. Then $\|u\|_\infty = \|T_u\|_{op} = \|\Lambda_t\|_{op} = C_t$ which means that $|T_u(a^\star s + b^\star)| = \|T_u\|_{op}\|a^\star s + b^\star\|_1$.

Solving the rocket car problem (3)

The condition $|T_u(a^\star s + b^\star)| = \|T_u\|_{op}\|a^\star s + b^\star\|_1$ means that

$$\left|\int_0^t (a^\star s + b^\star)u(s)\,\mathrm{d}s\right| = \|u\|_\infty \cdot \int_0^t |a^\star s + b^\star|\,\mathrm{d}s.$$

and can only be satisfied if

$$u(s) \;=\; \pm C_t \cdot \mathrm{sign}(a^\star s + b^\star).$$

Given a constraint $\|u\|_\infty \leq 1$, the shortest possible time is $T := \min\{t \geq 0 \mid C_t \leq 1\}$. Then there is a unique optimal control, and this control can only take the values ± 1 with at most one switch in sign. This is a special case of the bang-bang principle.

Convexity (continued)

Definition. Let V be a real vector space and let X be an arbitrary subset of V. A point $p \in X$ is called an **extremal point** of X if p is not an inner point of a line segment with endpoints in X. Thus if $p = (1-t)x_1 + tx_2$ with $x_i \in X$ implies $x_1 = x_2 = p$.

Note that the definition of an extreme point does not involve any topological concept. However, often topological methods have to be used to establish the existence of extreme points.

Krein-Milman Theorem (1940) Let V be a topological vector space on which $V^\star$ separates points. Let K be a compact and convex subset of V, and let E be the set of extreme points of K. Then $K = \overline{\mathrm{conv}(E)}$.

Even the finite-dimensional version of this theorem is not entirely trivial. It can be proved by induction on the dimension of V (see Karlheinz Spindler, Höhere Mathematik, p. 330).

Mark Grigorievich Krein (1907-1989)

David Pinhusovich Milman (1912-1982)

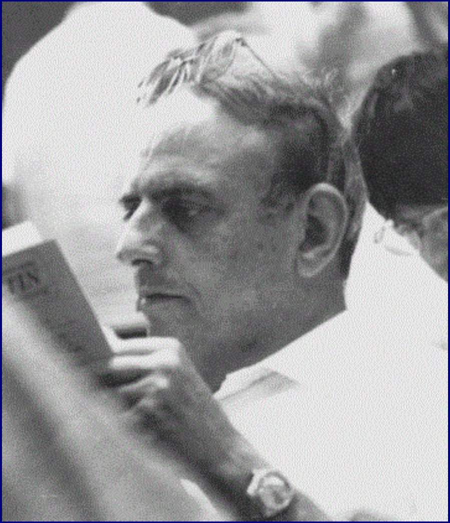
Convexity of the range of a vector measure

Let $\mathfrak{A}$ be a σ-algebra on a set X. A measure $\mu : \mathfrak{A} \to \mathbb{R}$ is called **atomless** if for any $A \in \mathfrak{A}$ with $\mu(A) > 0$ there is a subset $A_0 \subseteq A$ with $0 < \mu(A_0) < \mu(A)$.

Lyapunov's Convexity Theorem (1940) Let $\mu_1, \ldots, \mu_n$ be finite atomless measures on $(X, \mathfrak{A})$ and let $\mu = (\mu_1, \ldots, \mu_n)$. Then $\{\mu(A) \mid A \in \mathfrak{A}\}$ is a convex subset of $\mathbb{R}^n$.

The original proof of Lyapunov's theorem was both long and complicated. A dazzling short proof was given by Lindenstrauss in 1966 which relied heavily on functional analytic methods (Radon-Nikodym, Banach-Alaoglu, Krein-Milman). In the meantime, elementary proofs were found.

Aleksej Andreevich Lyapunov (1911-1973)



Joram Lindenstrauss (1936-2012)

Bang-bang principle

The following is a consequence of Lyapunov's theorem and the Krein-Milman theorem.

Theorem. Let $y_i : [a, b] \to \mathbb{R}$ be L^1-functions and let $y := (y_1, \ldots, y_n)$. Then

$$\begin{aligned} & \{\int_a^b y(t)u(t)\,\mathrm{d}t \mid |u_i(t)| \leq 1 \text{ for all } t\} \\ = \ & \{\int_a^b y(t)u(t)\,\mathrm{d}t \mid |u_i(t)| = 1 \text{ for all } t\}. \end{aligned}$$

This implies that if a time-optimal control for a linear system exists at all then there is also a "bang-bang" optimal control. This was first proved by LaSalle in 1959.

Joseph Pierre LaSalle (1916-1983) at RIAS