An Introduction to Functional Analysis through Applications

Abstracts of the lectures

Lecture 1: *The Historical Development of Functional Analysis*
Lecture 2: *From Linear Algebra and Calculus to Functional Analysis*
Lecturer: Karlheinz Spindler

The development of functional analysis at the beginning of the 20th century marks a move towards a more abstract approach to mathematics which was also felt in neighboring areas such as set-theoretical topology, integration theory and linear algebra and which led to a reappraisal of basic mathematical concepts (numbers, functions, foundations of geometry). However, higher abstraction was not a goal in itself; pursued by protagonists like Fredholm, Volterra, Hilbert, Riesz, Banach and others, it developed quite naturally from concrete problems in analysis and physics. The two talks will clarify this historical development and will also identify examples from undergraduate lectures such as analysis and linear algebra which lead to questions later pursued in functional analysis.

Lecture 3: *Reproducing Kernel Hilbert Spaces*
Lecture 4: *Kernel methods in Data Mining*
Lecturer: Hagen Knaf

The central objects of the two talks are Hilbert spaces H, whose elements are functions $f : X \to \mathbb{R}$ on a set X. Addition and scalar multiplication in H are the pointwise operations with these functions and the scalar product on H is required to give continous evaluation functionals $e_x : H \to \mathbb{R}$, $e_x(f) := f(x)$, for every point $x \in X$. These so-called Reproducing Kernel Hilbert Spaces (RKHS) are tightly connected to the class of positive semidefinite, symmetric functions $K : X \times X \to \mathbb{R}$, also called kernel functions. In the first talk the relationship between Reproducing Kernel Hilbert Spaces and kernel functions will be explained, culminating in the theorem of Aronszajn-Moore, that gives a complete overview over all RKHS.

An important feature of RKHS from the viewpoint of applications is the existence of an injective map $\Phi : X \to H$ with the property

$$\langle \Phi(x), \Phi(y) \rangle = K(x, y) \text{ for all } x, y \in X, \tag{1}$$

where $K : X \times X \to \mathbb{R}$ is a kernel function attached to H. In Data Mining this feature can be used to perform nonlinear data analysis tasks effectively: as an example consider a data set $X \subset \mathbb{R}^n$ subdivided into $r \geq 2$ disjoint groups. In Discriminant Analysis the task consists of determining hypersurfaces $S_1, \ldots, S_{r-1} \subset \mathbb{R}^n$ that separate the groups in an optimal way. Instead of solving a corresponding nonlinear optimisation problem in $\mathbb{R}^n$, one can first embed X via the map $\Phi : X \to H$ into a RKHS derived from a suitable kernel function $K : X \times X \to \mathbb{R}$, determine hyperplanes $H_1, \ldots, H_{r-1} \subset H$ that separate the groups in H, which can be done using linear techniques, and pull the result back to $\mathbb{R}^n$: $S_i := \Phi^{-1}(H_i)$. Due to the equation (1) in this procedure one actually never has to work in H but can stay in $\mathbb{R}^n$. In the second talk the details of the procedure just sketched are given and an example is discussed.

Lecture 5: *Functional Analysis and Partial Differential Equations*
Lecture 6: *Functional Analytic Foundations of Numerical Methods*
Lecturer: Thomas Lorenz

Partial differential equations occur in many applications in physics, chemistry, quantitative finance and the engineering sciences. However, explicit solution formulas are hardly available. Standard methods like the separation of variables, Green functions or integral transformations are usually restricted to very special cases in regard to the geometry or the equations.

Functional analysis offers a way out for some partial differential equations. The key idea is based on re-formulating the ≫classical≪ differential equation as an integral equation. In contrast to ordinary differential equations, we cannot apply the fundamental theorem of calculus here because the considered functions depend on more than one real variable. Hence, Gauß' theorem will take its role in the general setting – together with arbitrarily chosen test functions.

The integral problem with arbitrary test functions looks much more abstract at first glance, but it seizes many familiar ideas from linear algebra and adapts them to complete infinite-dimensional vector spaces with a scalar product (called Hilbert spaces). Pursuing these ideas, we derive the Lax-Milgram theorem which lays the basis for solving elliptic differential equations. This theorem is not just a theoretical tool, but opens the door to approximating the desired solution (which lives in an infinite-dimensional space) by a sequence of auxiliary solutions living in finite-dimensional spaces. Thus, it is the starting point for the well-established finite element methods, that are addressed in the next lecture.

Lecture 7: *Computer Workshop on Numerical Methods*
Lecturer: Alexander Ekhlakov

Mathematical models of engineering systems are often characterized by complex boundary value problems for partial differential equations posed in geometrically complicated regions. These problems are not amenable to an analytical treatment, but require numerical methods such as the finite element method, which turns out to be one of the most powerful ways to find approximative solutions of partial differential equations. It is based on the idea that every system is physically composed of different parts and hence its solutions may be also represented in parts. Thus the domain of the boundary value problem is divided into geometrically simpler subdomains, called finite elements, connected by a finite number of preselected points, called nodes. The unknown variables (for example, temperatures or displacements) over each finite element are approximated by linear combinations of basic functions and undetermined coefficients. Algebraic relations between these coefficients are obtained by governing equations, often in a weighted-integral sense, over each element. The unknown parameters represent the values of unknown variables at nodes. The algebraic relations from all elements are assembled using boundary conditions, continuity and equilibrium considerations.

While the previous lectures dealt with the functional analytic foundations of the finite element method, the computer workshop will give a “hands on” feeling for the practical use of this method. The main steps of the finite element method and their computer implementation will be demonstrated and applied to simple mechanical models.

Lectures 8+9: *Hilbert Space Methods in Quantum Mechanics*
Lecturer: Detlef Lehmann

In the first talk we review the basic physical facts which led to the discovery of quantum mechanics and we list the postulates of quantum mechanics.

In the second talk we apply these postulates to the hydrogen atom and we present the calculation of the eigenfunctions and eigenvalues of the Schroedinger equation in the presence of a Coulomb potential. This then leads to a full description of the observed energy spectrum of the hydrogen atom.

Lecture 10+11: *Martingales and Their Application in Mathematical Finance*
Lecturer: Claas Becker

Martingales are an important concept in modern probability theory.
A sequence (X_n) of random variables is called a martingale with respect to an increasing sequence $(\mathcal{A}_n)$ of sub-σ-fields if

$$E(X_m|\mathcal{A}_n) = X_n \quad \forall n \leq m \ .$$

$E(X|\mathcal{A})$ denotes the conditional expectation of a random variable X with respect to a sub-σ-field $\mathcal{A}$.
From a Hilbert space perspective, conditional expectations are orthogonal projections onto closed subspaces. This provides an elegant way of illustrating some properties of martingales. In mathematical finance, the price of any future payoff can be represented as a martingale.

Lecture 12: *Functional Analysis and Control Theory*
Lecturer: Karlheinz Spindler

One historical source of functional analysis was the calculus of variations, dealing with optimization problems in function spaces. Functional analysis helped to clarify the distinction between different types of extrema and the precise meaning of finding "best" solutions to variational problems. The modern counterpart of the calculus of variations, optimal control theory, deals with the problem of steering a system to a desired target state in an optimal way, and here also functional analysis plays an important role. The talk will lead up to and motivate Pontryagin's Maximum Principle, the crowning glory of 250 years of work in the calculus of variations and one of the highlights of 20th century mathematics. The lecture will be enhanced by computer demonstrations developed by 4th semester students in a programming project within an introduction to control theory.