What is functional analysis?

One possible answer: Functional analysis is the systematic study of coupled algebraic and topological structures.

The purpose of this talk is to sketch

- what this means,
- how such an approach can grow out of undergraduate mathematics,
- why such an approach is useful to solve problems in analysis.

The structure of the system of real numbers

The real numbers are known to us from the first semester on (or even from school). Explicitly or implicitly, we make use of three different structures on $\mathbb{R}$, as follows:

- **Algebraic structure:** We can add and multiply real numbers and hence can perform calculations. Equipped with the algebraic operations $+$ and $\cdot$, the real numbers form a field.
- **Order structure:** We can compare real numbers in size. Equipped with the ordering $\leq$, the real numbers form a totally ordered set.
- **Topological structure:** We can compare how close real numbers are, which sets form a neighborhood of a given point, and so on. This topological structure is embodied in the metric $d(x, y) = |x - y|$. Using this structure, we can talk about convergence and continuity.

Compatibility of the different structures

We work with these different structures simultaneously, exploiting (usually unconsciously) the fact that these structures are compatible with each other.

- The compatibility of the algebraic structure and the order structure is embodied in the **monotonicity laws**: If $a < b$ then $a + c < b + c$ for all $c \in \mathbb{R}$ and $ac < bc$ for all $c > 0$.
- The compatibility of the algebraic structure and the topological structure is embodied in the **continuity** of the algebraic operations: If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$ and $a_n b_n \to ab$.
- The compatibility of the order structure and the topological structure stems from the fact that the metric $d(x, y) = |x - y|$ is defined in terms of the absolut value which, in turn, is directly defined in terms of the order:

$$|x| := \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x \leq 0. \end{cases}$$

Generalization to several variables

The set $\mathbb{R}^n$ is the natural habitat for functions in several variables. This set carries various structures.

- **Algebraic structure:** We can add elements of $\mathbb{R}^n$ and multiply them by scalars, thereby making $\mathbb{R}^n$ into a real **vector space**.
- **Geometric structure:** We can introduce an inner product (or scalar product) $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, which allows us to talk about lengths of vectors and angles between vectors. This makes $\mathbb{R}^n$ into a **Euclidean space**.
- **Topological structure:** Denoting by $\|x\| := \sqrt{\langle x, x \rangle}$ the length of a vector $x \in \mathbb{R}^n$, we can define the distance $d(x, y) := \|x - y\|$ between two vectors. This metric yields a topological structure on $\mathbb{R}^n$. Using this structure, we can talk about convergence and continuity.

Topological vector spaces

Definition. A **topological vector space** over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a vector space over $\mathbb{K}$ together with a topology such that the addition

$$\begin{array}{ccc} V \times V & \to & V \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{array}$$

and the scalar multiplication

$$\begin{array}{ccc} \mathbb{K} \times V & \to & V \\ (\lambda, v) & \mapsto & \lambda v \end{array}$$

are continuous. Thus one requires that the vector space structure and the topological structure be compatible.

To a large extent, functional analysis is the study of topological vector spaces, but one also studies topological groups, topological rings, topological algebras, topological lattices and other mixed algebro-topological structures.

Norms and metrics (1)

Definition. A **norm** on a (real or complex) vector space V is a mapping $V \to \mathbb{R}$ denoted by $v \mapsto \|v\|$ which has the following properties:

- $\|v\| \geq 0$ for all $v \in V$ with equality if and only if $v = 0$;
- $\|\lambda v\| = |\lambda| \, \|v\|$ for all $v \in V$ and all $\lambda \in \mathbb{R}$;
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

We interpret $\|v\|$ as the "length" or "size" of the vector $v \in V$.

Definition. A **metric** on a set X is a mapping $d : X \times X \to \mathbb{R}$ which has the following properties:

- $d(x, y) \geq 0$ for all $x, y \in X$ with equality if and only if $x = y$;
- $d(y, x) = d(x, y)$ for all $x, y \in X$;
- $d(x, y) \leq d(x, u) + d(u, y)$ for all $x, y, u \in X$.

We interpret $d(x, y)$ as the "distance" between $x, y \in X$.

Norms and metrics (2)

Suppose we want to equip a (real or complex) vector space V with a metric d which respects the vector space operations in the following sense:

- $d(x + a, y + a) = d(x, y)$ for all $x, y, a \in V$ (**translation invariance**);
- $d(\lambda x, \lambda y) = |\lambda|\, d(x, y)$ for all $x, y \in V$ and all scalars λ (**homogeneity**).

How can we find such a metric?

Theorem. A metric $d : V \times V \to \mathbb{R}$ has the above properties if and only if it has the form $d(x, y) = \|x - y\|$ where $\| \cdot \|$ is a norm on V. (Proof as exercise!)

Normed spaces are topological vector spaces

Theorem. Every normed space is a topological vector space.

Proof. Let V be a normed space. We have to show that the addition and the scalar multiplication are continuous. For the addition, we have to show that if $v_n \to v$ and $w_n \to w$ in V implies that $v_n + w_n \to v + w$. This holds because

$$\begin{aligned} \|(v_n + w_n) - (v + w)\| &= \|v_n - v + w_n - w\| \\ &\leq \|v_n - v\| + \|w_n - w\|. \end{aligned}$$

For the scalar multiplication, we have to show that if $\lambda_n \to \lambda$ in $\mathbb{K}$ and $v_n \to v$ in V then $\lambda_n v_n \to \lambda v$. This holds because

$$\begin{aligned} &\|\lambda_n v_n - \lambda v\| = \|\lambda_n(v_n - v) + (\lambda_n - \lambda)v\| \\ &\leq \|\lambda_n(v_n - v)\| + \|(\lambda_n - \lambda)v\| \\ &= |\lambda_n| \cdot \|v_n - v\| + |\lambda_n - \lambda| \cdot \|v\|. \end{aligned}$$

Inner products and norms

Definition. An **inner product** (or **scalar product**) on a vector space V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a mapping $V \times V \to \mathbb{K}$ denoted by $(v, w) \mapsto \langle v, w \rangle$ which has the following properties:

- $\langle v, v \rangle \geq 0$ for all $v \in V$ with equality if and only if $v = 0$;
- $\langle w, v \rangle = \overline{\langle v, w \rangle}$ for all $v, w \in V$;
- $\langle \sum_{i=1}^{m} \lambda_i v_i, \sum_{j=1}^{n} \mu_j w_j \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \overline{\mu_j} \langle v_i, w_j \rangle$ for all vectors $v_i, w_j \in V$ and all scalars $\lambda_i, \mu_j \in \mathbb{K}$.

It is easy to check that if $\langle \cdot, \cdot \rangle$ is an inner product then $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm. How can one decide whether or not a given norm is induced by an inner product?

Theorem. A norm $\| \cdot \|$ on a vector space V over $\mathbb{K}$ is induced by an inner product if and only if for all $v, w \in V$ the **parallelogram law** $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$ holds.

Examples for norms (1)

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. There are many different ways to introduce norms on $\mathbb{K}^n$. The following are the most common ones:

- $\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$;
- $\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|$;
- $\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.

More generally, we can introduce the norms

$$\|x\|_p := \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}$$

for all $p \geq 1$, for which $\|x\|_\infty$ is obtained as the limit for $p \to \infty$. (Of these norms, only $\|\cdot\|_2$ is derived from an inner product.) Many other norms on $\mathbb{K}^n$ are possible. For example, if $\|\cdot\|$ is any norm on $\mathbb{K}^n$ and if $A \in \mathbb{K}^{n\times n}$ is any invertible matrix then $|||x||| := \|Ax\|$ is again a norm on $\mathbb{K}^n$.

If V is any finite-dimensional vector space over $\mathbb{K}$, we can identify V with $\mathbb{K}^n$ by choosing a basis, thereby identifying norms on V with norms on $\mathbb{K}^n$. Hence V can be equipped with a variety of different norms.

Examples for norms (2)

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $C[a,b]$ be the space of all continuous functions $x : [a,b] \to \mathbb{K}$. We can equip $C[a,b]$ with various different norms, for example the following ones:

- $\|x\|_2 := \sqrt{\int_a^b |x(t)|^2 \mathrm{d}t}$;
- $\|x\|_1 := \int_a^b |x(t)| \, \mathrm{d}t$;
- $\|x\|_\infty := \max\{|x(t)| \mid a \leq t \leq b\}$.

More generally, we can introduce the norms

$$\|x\|_p := \sqrt[p]{\int_a^b |x(t)|^p \mathrm{d}t}$$

for all $p \geq 1$. Note the analogy with the finite-dimensional case, which becomes clear if we interpret a function $x : [a,b] \to \mathbb{K}$ as a vector $\big(x(t)\big)_{a \leq t \leq b}$ with an infinite number of components $x(t)$, interpreting the argument t as an index.

Equivalence of norms

Definition. Two norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ on a (real or complex) vector space V are called **equivalent** if there are constants $c > 0$ and $C > 0$ such that

$$c\|x\| \leq |\!|\!|x|\!|\!| \leq C\|x\| \quad \text{for all } x \in V.$$

It is easy to see that this is indeed an equivalence relation on the set of all norms on V. Moreover, the following holds true. (Exercise!)

Theorem. Two norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ on V are equivalent if and only if, given a sequence (x_n) in V and an element $x \in V$, the condition $\|x_n - x\| \to 0$ holds if and only if the condition $|\!|\!|x_n - x|\!|\!| \to 0$ holds.

Thus as far as topological properties (convergence, continuity, ...) are concerned, we do not have to distinguish between equivalent norms.

Examples for equivalent norms

The norms $\|\cdot\|_p$ on $\mathbb{K}^n$ (where $1 \leq p \leq \infty$) are all equivalent. Assume $1 \leq p < \infty$. Given $x \in \mathbb{K}^n$, we have

$$\begin{aligned} \|x\|_\infty &= \max_{1\leq i\leq n} |x_i| = |x_{i_0}| \\ &= \sqrt[p]{0^p + \cdots + |x_{i_0}|^p + \cdots + 0^p} \\ &\leq \sqrt[p]{|x_1|^p + \cdots + |x_{i_0}|^p + \cdots + |x_n|^p} = \|x\|_p. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|x\|_p &= \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p} \\ &\leq \sqrt[p]{\|x\|_\infty^p + \|x\|_\infty^p + \cdots + \|x\|_\infty^p} \\ &= \sqrt[p]{n \cdot \|x\|_\infty^p} = \sqrt[p]{n} \cdot \|x\|_\infty. \end{aligned}$$

Hence $\|x\|_\infty \leq \|x\|_p \leq \sqrt[p]{n} \cdot \|x\|_\infty$ for all $x \in \mathbb{K}^n$, so that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalent. Two norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are thus both equivalent to $\|\cdot\|_\infty$ and consequently also equivalent to each other.

Norms on finite-dimensional vector spaces (1)

Theorem. Any two norms on a finite-dimensional $\mathbb{K}$-vector space V are equivalent.

Proof. Let $\|\cdot\|$ be any norm on V. Choose a basis $(b_1, \ldots, b_n)$ of V and consider the reference norm $|||x_1 b_1 + \cdots + x_n b_n||| := |x_1| + \cdots + |x_n|$. Letting $C := \max(\|b_1\|, \ldots, \|b_n\|)$, we have $\|x\| \leq |x_1|\,\|b_1\| + \cdots + |x_n|\,\|b_n\| \leq C|||x|||$ for all $x \in V$. Consequently, $\|\cdot\| : V \to \mathbb{R}$ is continuous with respect to $|||\cdot|||$ because $|\,\|x\| - \|y\|\,| \leq \|x - y\| \leq C \cdot |||x - y|||$ for all $x, y \in V$. Hence $\|\cdot\|$ takes its minimum on the compact set $S := \{x \in V \mid |||x||| = 1\}$, which shows that there is a number $c > 0$ such that $\|x\| \geq c$ whenever $|||x||| = 1$. Thus if $x \neq 0$ then

$$\left\| \frac{x}{|||x|||} \right\| \geq c$$

for all $x \neq 0$ and hence $\|x\| \geq c|||x|||$ for all $x \in V$. The inequality $c|||x||| \leq \|x\| \leq C|||x|||$ shows that the arbitrary norm $\|\cdot\|$ is equivalent to the reference norm $|||\cdot|||$.

Norms on finite-dimensional vector spaces (2)

If we choose a basis $(b_1, \ldots, b_n)$ of V and write each vector $x \in V$ as $x = x_1 b_1 + \cdots + x_n b_n$, then $x^{(k)} \to x$ in V if and only if $x_i^{(k)} \to x_i$ in $\mathbb{K}$ for all indices $1 \leq i \leq n$. This can be expressed by saying that the following two conditions are equivalent:

- $x^{(k)} \to x$ (strong convergence);
- $\langle x^{(k)}, a \rangle \to \langle x, a \rangle$ for all $a \in V$ (weak convergence).

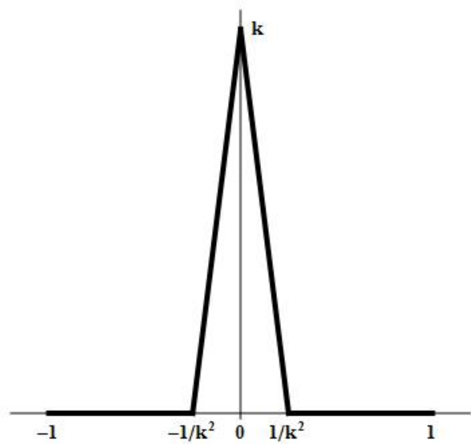
A finite-dimensional vector space V over $\mathbb{K}$ carries a unique topology which makes it into a topological vector space. This topology is induced by any norm on V. Convergence with respect to this topology is coordinatewise convergence with respect to any chosen basis of V.

Examples of inequivalent norms

Are the following norms on $C[-1,1]$ equivalent?

- $\|f\|_\infty := \max\limits_{-1\le x\le 1} |f(x)|$
- $\|f\|_1 := \int_{-1}^{1} |f(x)|\,\mathrm{d}x$

Clearly, $\|f\|_1 \le 2\|f\|_\infty$ for all f. On the other hand, there can be no constant C such that $\|f\|_\infty \le C\|f\|_1$. For example, we can find a sequence (f_k) of functions with $\|f_k\|_1 \to 0$ and $\|f_k\|_\infty \to \infty$ for $k \to \infty$.



Exercise

Given an interval $I = [a, b]$, we denote by $C^1(I)$ the space of all continuously differentiable functions $f : I \to \mathbb{R}$. Which of the following norms on $C^1(I)$ are equivalent?

- $\|f\| := \max\limits_{a \le x \le b} |f(x)|$
- $\|f\| := |f(a)| + \max\limits_{a \le x \le b} |f'(x)|$
- $\|f\| := \max\limits_{a \le x \le b} |f(x)| + \max\limits_{a \le x \le b} |f'(x)|$
- $\|f\| := \int_a^b |f(x)| \, dx + \max\limits_{a \le x \le b} |f'(x)|$
- $\|f\| := |f(a)| + \int_a^b |f'(x)| \, dx$

Examples of infinite-dimensional normed spaces

- $\ell^p := \{(a_1, a_2, a_3, \dots) \mid \|a\|_p := \sqrt[p]{\sum_{k=1}^{\infty} |a_k|^p} < \infty\}$
- $\ell^\infty := \{(a_1, a_2, a_3, \dots) \mid \|a\|_\infty := \sup_{k \geq 1} |a_k| < \infty\}$
- $L^p(I) := \{f : I \to \mathbb{K} \mid \|f\|_p := \sqrt[p]{\int_I |f(x)|^p \mathrm{d}x} < \infty\}$
- $L^\infty(I) := \{f : I \to \mathbb{K} \mid \|f\|_\infty := \operatorname{ess\,sup}_{x \in I} |f(x)| < \infty\}$
- $C_b(I) := \{f : I \to \mathbb{K} \mid f \text{ continuous}, \ \|f\|_\infty := \sup_{x \in I} |f(x)| < \infty\}$

Exercise. Let $a = (a_1, a_2, a_3, \dots) \in \ell^p$ where $1 \leq p \leq \infty$ and let

$$a^{(n)} := (a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ 0,\ 0,\ 0,\ \dots).$$

When does $a^{(n)} \to a$ hold in ℓ^p?

Geometry of inner product spaces

Both from a theoretical viewpoint and from the viewpoint of applications, inner product spaces are "better" than general normed spaces. This is because an inner product, as compared to an arbitrary norm, provides us with an additional geometric structure. In particular, we call two elements $v_1, v_2 \in V$ of an inner product space V **orthogonal** and write $v_1 \perp v_2$, if $\langle v_1, v_2 \rangle = 0$. This gives rise to concepts like

- orthonormal system,
- orthonormal basis,
- orthogonal projection,
- orthogonal complement.

To a large extent, geometric intuition from the two- and three-dimensional situation can be transferred to infinite-dimensional spaces.

Best approximation and orthogonality (1)

Theorem. Let V be an inner product space, U a subspace of V and $x \in V$ an arbitrary element of V. Given a vector $u_0 \in U$, the following two conditions are equivalent:

- $\|x - u_0\| \leq \|x - u\|$ for all $u \in U$;
- $x - u_0 \in U^\perp$ (i.e., $x - u_0 \perp u$ for all $u \in U$).

Proof. Assume that $\|x - u_0\| \leq \|x - u\|$ for all $u \in U$. Then for all $u \in U$ and all $\lambda > 0$ we have

$$\begin{aligned} \|x - u_0\|^2 &\leq \|x - u_0 \pm \lambda u\|^2 \\ &= \|x - u_0\|^2 + 2\mathrm{Re}\langle x - u_0, \pm\lambda u\rangle + \| \pm \lambda u\|^2 \\ &= \|x - u_0\|^2 \pm 2\lambda \mathrm{Re}\langle x - u_0, u\rangle + \lambda^2\|u\|^2, \end{aligned}$$

hence $\mp 2\lambda \mathrm{Re}\langle x - u_0, u\rangle \leq \lambda^2\|u\|^2$, i.e., $2|\mathrm{Re}\langle x - u_0, u\rangle| \leq \lambda\|u\|^2$. Since $\lambda > 0$ may be arbitrarily small, this implies that $\mathrm{Re}\langle x - u_0, u\rangle = 0$. Since $u \in U$ was arbitrary, we have $\mathrm{Re}\langle x - u_0, u\rangle = 0$ for all $u \in U$. For $\mathbb{K} = \mathbb{R}$ this is the claim $x - u_0 \perp u$ already.

Best approximation and orthogonality (2)

(Continuation of Proof.) For $\mathbb{K} = \mathbb{C}$ we can replace u by iu; hence for all $u \in U$ we also have $0 = \mathrm{Re}\langle x - u_0, iu\rangle = \mathrm{Re}\big(-i\langle x - u_0, u\rangle\big) = \mathrm{Im}\langle x - u_0, u\rangle$ and therefore $\langle x - u_0, u\rangle = 0$.

Suppose conversely that $x - u_0 \perp U$. Given $u \in U$, we then have $\langle x - u_0, u_0 - u\rangle = 0$ and therefore

$$\begin{aligned} \|x-u\|^2 &= \|(x-u_0)+(u_0-u)\|^2 \\ &= \|x-u_0\|^2 + 2\mathrm{Re}\langle x-u_0, u_0-u\rangle + \|u_0-u\|^2 \\ &= \|x-u_0\|^2 + \|u_0-u\|^2 \;\geq\; \|x-u_0\|^2 \end{aligned}$$

with equality if and only if $u = u_0$.

Remark. Note that the above theorem says nothing about the *existence* of an element $u_0 \in U$ with the indicated properties.

Best approximation and orthogonality (3)

In fact, a best approximation of an element $x \in V$ in a subspace U need not exist.

Example. Let V be the space of all sequences (a_n) in $\mathbb{K}$ such that $a_n \neq 0$ only for a finite number of indices n. Define an inner product on V by letting $\langle a, b \rangle := \sum_n a_n \overline{b_n}$. Consider the subspace U consisting of all sequences $a = (a_n)$ in V with $\sum_n a_n = 0$. Let $x := (1, 0, 0, \ldots)$; we want to show that there is no best approximation of x in U. Define $u_n \in U$ by

$$u_n := (1, \underbrace{-\frac{1}{n}, \ldots, -\frac{1}{n}}_{n \text{ times}}, 0, 0, \ldots).$$

Then $\|x - u_n\|^2 = n \cdot (1/n^2) = 1/n$. Hence a best approximation $u_0 \in U$ for x would have to satisfy $\|x - u_0\| \leq \|x - u_n\| = 1/\sqrt{n}$ for all $n \in \mathbb{N}$ and hence $\|x - u_0\| = 0$, thus $u_0 = x$, contradicting the fact that $x \notin U$. **Exercise:** Verify directly that $U^\perp = \{0\}$.

A technical lemma

Lemma. If $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}$ are complex numbers then

$$\lambda\,\overline{\lambda} - 2\mathrm{Re}(\overline{\lambda}\,z) \geq -|z|^2$$

with equality if and only if $\lambda = z$.

Proof. Writing $z = a + ib$ and $\lambda = u + iv$ we obtain

$$\begin{aligned} & \lambda\,\overline{\lambda} - 2\mathrm{Re}(\overline{\lambda}\,z) \\ = \;& u^2 + v^2 - 2(au + bv) \\ = \;& u^2 - 2au + v^2 - 2bv \\ = \;& (u-a)^2 - a^2 + (v - b^2) - b^2 \\ = \;& (u-a)^2 + (v-b)^2 - |z|^2 \;\geq\; -|z|^2 \end{aligned}$$

with equality if and only if $u = a$ and $v = b$.

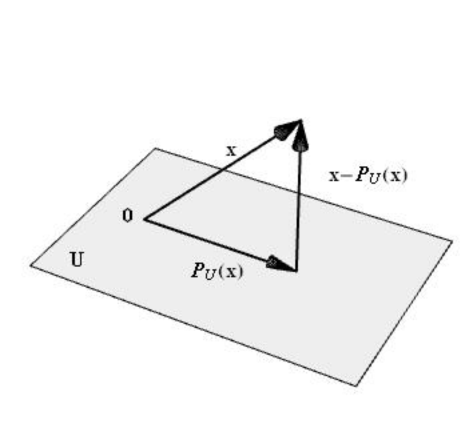
Orthogonal projection (1)

Let V be an inner product space, U a finite-dimensional subspace of V and $x \in V$ an arbitrary element of V. Then there is a unique element $u_0 \in U$ with

$$\|x - u_0\| \leq \|x - u\|$$

for all $u \in U$. This element is called the **orthogonal projection** of x onto U and is denoted by $P_U(x)$. If $(e_1, \ldots, e_n)$ is any orthonormal basis of U then

$$P_U(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$



Orthogonal projection (2)

Proof. Choose an orthonormal basis $(e_1, \ldots, e_n)$ of U (for example by applying the Gram-Schmidt procedure). Each element $u \in U$ possesses a unique representation $u = \sum_{i=1}^{n} \lambda_i e_i$. It follows that

$$
\begin{aligned}
\|x-u\|^2 &= \|x - \sum_{i=1}^{n} \lambda_i e_i\|^2 \\
&= \|x\|^2 - 2\mathrm{Re}\langle x, \sum_{i=1}^{n} \lambda_i e_i\rangle + \langle \sum_{i=1}^{n} \lambda_i e_i, \sum_{j=1}^{n} \lambda_j e_j\rangle \\
&= \|x\|^2 - 2\sum_{i=1}^{n} \mathrm{Re}\big(\overline{\lambda_i}\langle x, e_i\rangle\big) + \sum_{i=1}^{n} \lambda_i \overline{\lambda_i} \\
&= \|x\|^2 + \sum_{i=1}^{n} \left(\lambda_i \overline{\lambda_i} - 2\mathrm{Re}\big(\overline{\lambda_i}\langle x, e_i\rangle\big)\right)
\end{aligned}
$$

Applying our technical lemma with $\lambda = \lambda_i$ and $z = \langle x, e_i\rangle$, we see that $\|x-u\|^2 \geq \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$ with equality if and only if $\lambda_i = \langle x, e_i\rangle$ for $1 \leq i \leq n$.

Exercises

Exercise. Find $\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \mathrm{d}x$ and maximize $\int_{-1}^{1} x^3 g(x)\,\mathrm{d}x$ subject to the conditions $\int_{-1}^{1} |g(x)|^2 \mathrm{d}x = 1$ and

$$0 = \int_{-1}^{1} g(x)\,\mathrm{d}x = \int_{-1}^{1} x g(x)\,\mathrm{d}x = \int_{-1}^{1} x^2 g(x)\,\mathrm{d}x.$$

Exercise. Find $\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 e^{-x} \mathrm{d}x$. Moreover, formulate and solve the corresponding maximum problem as in the previous exercise.

A static versus a dynamic point of view

Static point of view:

- Solve the algebraic equation $x^3 - 3x + 1 = 0$.
- Solve a system $Ax = b$ of linear equations.
- Solve the differential equation $y''(x) + ay'(x) + by(x) = f(x)$.
- Solve the integral equation $f(x) = \int_a^b K(x, y)u(y)\mathrm{d}y$.

Dynamic point of view:

- Study the function $x \mapsto x^3 - 3x + 1$.
- Study the linear mapping $x \mapsto Ax$.
- Study the differential operator $y \mapsto y'' + ay' + by$.
- Study the integral operator $u \mapsto \int_a^b K(\cdot, y)u(y)\mathrm{d}y$.

Conclusion: Along with elements of spaces carrying a certain structure, also study structure-preserving mappings between these spaces. To do so, consider spaces of structure-preserving mappings themselves as structured spaces.

Linear mappings on infinite-dimensional vector spaces

In an infinite-dimensional setting, linear mappings can behave differently than in a finite-dimensional setting (which is typically studied in Linear Algebra).

- If $A, B : V \to V$ are endomorphisms of a finite-dimensional space V it may happen that $AB \neq BA$. However, if $AB = \mathbf{1}$ then also $BA = \mathbf{1}$ so that left-invertibility, right-invertibility and invertibility coincide. This is no longer true if V is infinite-dimensional.
- If $A : V \to V$ is an endomorphism of a finite-dimensional vector space over K and if $\lambda \in K$ then $A - \lambda\mathbf{1}$ is not injective if and only if $A - \lambda\mathbf{1}$ is not surjective if and only if $\det(A - \lambda\mathbf{1}) = 0$. If V is infinite-dimensional then the third condition does not make sense, and the first two conditions are not equivalent in general. Hence spectral theory becomes more complicated in an infinite-dimensional setting (and is of extreme importance in the study of differential and integral equations).

Spectrum of linear operators (1)

Define $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $Df = f'$. Let us determine the image and the kernel of $D - \lambda\,\mathbf{1}$. First,

$$f \in \ker(D - \lambda\,\mathrm{id}) \;\Leftrightarrow\; f' - \lambda f = 0 \;\Leftrightarrow\; f(x) = Ce^{\lambda x}.$$

Hence **every** scalar $\lambda \in \mathbb{K}$ is an eigenvalue of D (with a one-dimensional eigenspace). Next, $F \in \mathrm{im}(D - \lambda\,\mathrm{id})$ if and only if there is a function $f \in C^\infty(\mathbb{R})$ such that

$$F = f' - \lambda f \quad\Leftrightarrow\quad f(x) = Ce^{\lambda x} + \int_0^x e^{\lambda(x-\xi)} F(\xi)\,\mathrm{d}\xi.$$

Hence $D - \lambda\,\mathrm{id}$ is onto (surjective) for **every** scalar $\lambda \in \mathbb{K}$.

Spectrum of linear operators (2)

Define $I : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $(If)(x) := \int_0^x f$. Let us determine the image and the kernel of $I - \lambda\,\mathrm{id}$. First, $f \in \ker(I - \lambda\,\mathrm{id})$ if and only if

$$If = \lambda f' \;\Leftrightarrow\; \int_0^x f = \lambda f(x) \;\Leftrightarrow\; f = \lambda f' \text{ and } \lambda f(0) = 0 \;\Leftrightarrow\; f = 0.$$

Hence $I - \lambda\,\mathrm{id}$ is one-to-one (injective) for every $\lambda \in \mathbb{K}$. Hence **no** scalar $\lambda \in \mathbb{K}$ is an eigenvalue of I. Next, $F \in \mathrm{im}(I - \lambda\,\mathrm{id})$ if and only if there is a function $f \in C^\infty(\mathbb{R})$ such that $F = If - \lambda f$, i.e.,

$$F(x) = \int_0^x f - \lambda\, f(x) \;\Leftrightarrow\; F' = f - \lambda f' \text{ and } F(0) = -\lambda f(0).$$

For $\lambda = 0$ this is the case if and only if $F(0) = 0$. For $\lambda \neq 0$ this is the case if and only if $f(x) = -\lambda^{-1} e^{x/\lambda} \left[F(0) + \int_0^x F'(\xi) e^{-\xi/\lambda}\,\mathrm{d}\xi\right] = -\lambda^{-1} F(x) - \lambda^{-2} e^{-\lambda/x} \int_0^x F(\xi) e^{-\xi/\lambda}\,\mathrm{d}\xi$. Hence $I - \lambda\,\mathrm{id}$ is surjective for all $\lambda \neq 0$, but not for $\lambda = 0$.

Spectrum of linear operators (3)

Let V be the K-vector space of all sequences $(a_1, a_2, a_3, \dots)$ in K. Define the **right shift** $R : V \to V$ by

$$R : (a_1, a_2, a_3, a_4, \dots) \;\mapsto\; (0, a_1, a_2, a_3, \dots).$$

Given $\lambda \in K$, we want to investigate the solvability of the equation $(R - \lambda\,\mathrm{id})(x) = y$, i.e.,

$$(-\lambda x_1,\, x_1 - \lambda x_2,\, x_2 - \lambda x_3, \dots) \;=\; (y_1,\, y_2,\, y_3,\, \dots).$$

If $\lambda \neq 0$ then there is a unique solution x for any given y (namely $x \in V$ where $x_1 = -y_1/\lambda$ and $x_k = (x_{k-1} - y_k)/\lambda$ for $k \geq 2$). Hence $R - \lambda\,\mathrm{id}$ is a bijection if $\lambda \neq 0$. If $\lambda = 0$ then $R - \lambda\,\mathrm{id}$ is one-to-one, but not onto, the image of $(R - \lambda\,\mathrm{id})$ being the subspace of all sequences whose first entry is zero. This subspace has codimension one.

Spectrum of linear operators (4)

Similarly, we can define the **left shift** $L : V \to V$ by

$$L : (a_1, a_2, a_3, a_4, \dots) \;\mapsto\; (a_2, a_3, a_4, a_5, \dots).$$

Given $\lambda \in K$, we want to investigate the solvability of the equation $(L - \lambda\,\mathrm{id})(x) = y$, i.e.,

$$(x_2 - \lambda x_1,\, x_3 - \lambda x_2,\, x_4 - \lambda x_3, \dots) \;=\; (y_1,\, y_2,\, y_3,\, \dots).$$

Given $y \in V$, we can choose $x_1 = \xi$ arbitrarily and must then have $x_{k+1} = y_k + \lambda x_k = y_k + \lambda^k x_1$. Thus the equation $(L - \lambda\,\mathrm{id})(x) = y$ has the general solution

$$(x_1, x_2, x_3, \dots) = (0, y_1, y_2, y_3, \dots) + \xi\,(1, \lambda, \lambda^2, \lambda^3, \dots)$$

where $\xi \in K$ is arbitrary. We see that, given any number $\lambda \in K$, the mapping $L - \lambda\,\mathrm{id}$ is surjective, but not injective. Every number $\lambda \in K$ is an eigenvalue of L, with a one-dimensional eigenspace.

Spectrum of linear operators: Exercises

Exercise. Consider the left shift L and the right shift R on the vector space of all sequences

$$(\ldots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots).$$

For which $\lambda \in K$ does $L - \lambda\,\mathrm{id}$ resp. $R - \lambda\,\mathrm{id}$ fail to be surjective or injective?

Exercise. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let V be the $\mathbb{K}$-vector space of all sequences in $\mathbb{K}$. Define $M : V \to V$ by

$$M(a_1, a_2, a_3, \ldots) \;:=\; \left(a_1,\, \frac{a_1 + a_2}{2},\, \frac{a_1 + a_2 + a_3}{3},\, \ldots\right).$$

For which $\lambda \in \mathbb{K}$ does $M - \lambda\,\mathrm{id}$ fail to be surjective or injective? Show that M leaves invariant the subspace U of all convergent sequences. Answer the same question as before for the restriction $M_U : U \to U$.

Continuous linear mappings

A linear mapping $T : X \to Y$ between normed spaces is called bounded if there is a constant $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. For a linear mapping $T : X \to Y$, the following conditions are equivalent:

(1) T is bounded;

(2) T is Lipschitz-continuous;

(3) T is continuous;

(4) T is continuous at 0.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. Assume (4) holds. Then for $\varepsilon := 1$ there is $\delta > 0$ such that if $\|x - 0\| \leq \delta$ then $\|T(x) - T(0)\| \leq \varepsilon$. Thus if $\|x\| \leq \delta$ then $\|Tx\| \leq 1$. Hence if $x \neq 0$ then

$$\left\| T\left(\frac{\delta x}{\|x\|}\right) \right\| \leq 1$$

which means that $\|Tx\| \leq (1/\delta)\,\|x\|$. This last equation also holds if $x = 0$. Thus T is bounded.

Spaces of operators

The set $\mathbb{B}(X, Y)$ of all bounded linear mappings is a vector space and, in fact, a normed space with the **operator norm**

$$\begin{aligned}
\|T\|_{\mathrm{op}} &:= \inf\{C \geq 0 \mid \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \\
&= \inf\{C \geq 0 \mid \|T(x/\|x\|)\| \leq C \text{ for all } x \neq 0\} \\
&= \inf\{C \geq 0 \mid \|Tx\| \leq C \text{ whenever } \|x\| = 1\} \\
&= \inf\{C \geq 0 \mid \|Tx\| \leq C \text{ whenever } \|x\| \leq 1\} \\
&= \sup\{\|Tx\| \mid \|x\| \leq 1\}.
\end{aligned}$$

Interpretation: $\|T\|_{\mathrm{op}}$ is the maximal stretching factor which the application of T can effect.

Important special case: The **dual space** $V^\star$ of a normed space V is the space of all continuous linear forms $f : V \to \mathbb{K}$.

Linear forms (1)

A linear mapping $f : \mathbb{K}^n \to \mathbb{K}$ is necessarily of the form

$$f(x) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

with numbers $a_i \in \mathbb{K}$. This can be written as $f(x) = \langle x, \overline{a} \rangle$. Hence each linear form on $\mathbb{K}^n$ is the inner product with a unique element of $\mathbb{K}^n$.

It is a natural question to ask whether a similar statement holds for all inner product spaces. The answer cannot be affirmative in general, because any functional of the form $\langle \cdot, v \rangle$ is necessarily continuous:

$$|\langle x, v \rangle - \langle y, v \rangle| = |\langle x - y, v \rangle| \leq \|x - y\| \, \|v\|.$$

But even for **continuous** linear functionals the answer is in general negative, as the following example shows.

Linear forms (2)

Example. Let V be the set of all sequences $x \in \ell^2(\mathbb{R})$ for which only a finite number of entries is nonzero. A linear form on V is given by

$$f(a) := a_1 + a_2 + a_3 + \cdots .$$

Then f is not continuous (why not?), hence cannot be of the form $\langle \cdot, v \rangle$ with some $v \in V$. Another linear form on V (this times continuous) is given by

$$f(a) := 1 \cdot a_1 + \frac{1}{2} \cdot a_2 + \frac{1}{3} \cdot a_3 + \cdots .$$

But again, there is no vector $v \in V$ such that $f = \langle \cdot, v \rangle$. The reason is that the space V is not complete. We will have to discuss the concept of completeness.

Completeness

Definition. A sequence $(x_1, x_2, x_3, \ldots)$ in a metric space (X, d) is called a **Cauchy sequence** if $d(x_m, x_n) \to 0$ as $m, n \to \infty$. The metric space (X, d) is called **complete** if every Cauchy sequence in X converges in X.

Completeness is of fundamental importance in analysis and numerical mathematics. For example, suppose you want to find the solution of an (algebraic, differential, integral) equation. Often it is possible to construct a sequence $(x_1, x_2, x_3, \ldots)$ of approximate solutions which (one hopes) will converge to a true solution. If the approximants x_k lie closer and closer together (i.e., form a Cauchy sequence), completeness guarantees the existence of $x := \lim_{k\to\infty} x_k$.

Definition. A complete normed space is called a **Banach space**. A complete inner product space is called a **Hilbert space**.

Stefan Banach
(1892-1945)

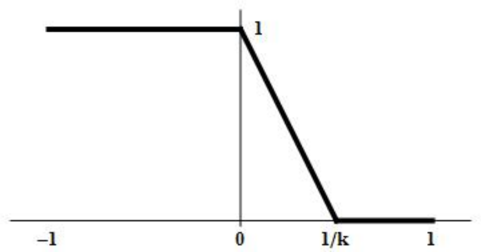
David Hilbert
(1862-1943)

Examples for complete spaces

- Every finite-dimensional normed space is complete. This is a consequence of the completeness of $\mathbb{R}$ and $\mathbb{C}$ and the fact that convergence in $\mathbb{K}^n$ is componentwise convergence.
- The space $C[a, b]$ with the norm $\|\cdot\|_\infty$ is complete. To see this, let (f_k) be a Cauchy sequence in $C[a, b]$. Then for each $x \in [a, b]$, the sequence $\big(f_k(x)\big)$ is a Cauchy sequence in $\mathbb{K}$, hence convergent; say $f_k(x) \to f(x)$. One checks that $f_k \to f$ not only pointwise, but even uniformly on $[a, b]$. But the uniform limit of a sequence of continuous functions is itself continuous. Hence $f \in C[a, b]$, and $f_k \to f$ in $C[a, b]$ (which means that $\|f_k - f\| \to 0$).

Examples for incomplete spaces

- Let V be the space of all real sequences $a = (a_1, a_2, a_3, \ldots)$ such that $a_i \neq 0$ for only finitely many indices i. An inner product on V is defined by $\langle a, b \rangle := \sum_{k=1}^{\infty} a_k b_k$. Let $a^{(n)} := (1,\ 1/2,\ 1/3,\ \ldots,\ 1/n, 0, 0, 0, \ldots)$. Then $(a^{(n)})$ is a Cauchy sequence in V which does not converge in V.
- The space $C[-1,1]$ with the norm $\|\cdot\|_1$ is not complete. On fact, the functions (f_k) defined below form a Cauchy sequence in $C[-1,1]$ which does not converge in $C[-1,1]$.



Completeness of dual spaces

Theorem. Let V be a normed space. Then $V^\star$ is complete.

Proof. Let (f_k) be a Cauchy sequence in V. Fix $v \in V$. Then $\big(f_k(v)\big)$ is a Cauchy sequence in $\mathbb{K}$, hence convergent. Let

$$f(v) := \lim_{k\to\infty} f_k(v).$$

Clearly, f is linear. Given $\varepsilon > 0$, there is a number $N \in \mathbb{N}$ such that $\|f_n - f_m\| \le \varepsilon$ for all $m, n \ge N$ so that

$$\begin{aligned}
& |f_n(v) - f_m(v)| \le \varepsilon \text{ for all } m, n \ge N \text{ and all } \|v\| \le 1 \\
\Rightarrow\quad & |f_n(v) - f(v)| \le \varepsilon \text{ for all } n \ge N \text{ and all } \|v\| \le 1 \\
\Rightarrow\quad & \|f_n - f\| \le \varepsilon \text{ for all } n \ge N \\
\Rightarrow\quad & \|f\| \le \|f - f_N\| + \|f_N\| \le \varepsilon + \|f_N\| < \infty.
\end{aligned}$$

Hence f is bounded, i.e., is an element of $V^\star$, and $f_n \to f$ in $V^\star$.

Completeness and closedness

Theorem. Let (X, d) be a complete metric space. Then a subspace $A \subseteq X$ (i.e., a subset of X equipped with the induced metric) is complete if and only if it is closed in X.

Proof. Let A be closed, and let (a_k) be a Cauchy sequence in A. Then (a_k) is a Cauchy sequence in X, hence convergent in X; say $a_k \to x$. Then $x \in \overline{A} = A$, so that (a_k) is convergent not only in X, but even in A. Hence A is complete.

Conversely, assume that A is complete. Assume that $x \in \overline{A}$, say $a_k \to x$ with a sequence (a_k) in A. Then (a_k) is a Cauchy sequence in A, hence convergent in A, say $a_k \to a$. Since a sequence cannot possess two different limits, we have $x = a \in A$. Since $x \in \overline{A}$ was arbitrary, we have $\overline{A} = A$. Hence A is closed.

Orthogonal complements

Theorem. Let U be a closed subspace of a Hilbert space. Then $V = U \oplus U^{\perp}$.

Proof (Riesz 1934). Let $x \in V \setminus U$ and $d := \inf\{\|x-u\| \mid u \in U\}$; then $d > 0$. There is a sequence (u_n) in U with $\|x-u_n\| \to d$. Now

$$\begin{aligned}
4d^2 + \|u_m - u_n\|^2 &\leq 4\|x - (u_m+u_n)/2\|^2 + \|u_m - u_n\|^2 \\
&= \|2x - (u_m+u_n)\|^2 + \|u_m - u_n\|^2 \\
&= \|(x-u_m) + (x-u_n)\|^2 + \|(x-u_m) - (x-u_n)\|^2 \\
&= 2\|x - u_m\|^2 + 2\|x - u_n\|^2 \to 4d^2 \text{ as } m, n \to \infty
\end{aligned}$$

where we used the parallelogram equation in the step from the third to the fourth line. Hence $\|u_m - u_n\| \to 0$ so that (u_n) is a Cauchy sequence, thus convergent because U is closed in V and hence complete. Assume $u_n \to u_0$; then $d = \|x - u_0\|$ so that u_0 is a best approximation of x in U.

The Riesz representation theorem for Hilbert spaces

Theorem. Let V be a Hilbert space and let $f \in V^\star$. Then there is a unique vector $v \in V$ such that $f = \langle \cdot, v \rangle$.

Proof. The uniqueness is clear because if $\langle x, v_1 \rangle = \langle x, v_2 \rangle$ for all $x \in V$ then necessarily $v_1 = v_2$. (Take $x := v_1 - v_2$.) Hence only the existence of v must be shown. If $f = 0$ then $v = 0$. Let $f \neq 0$. Then $U := \ker(f)$ is a closed subspace, hence possesses an orthogonal complement which is necessarily one-dimensional, say $U^\perp = \mathbb{K}\xi$ where $\|\xi\| = 1$. Given $x = u + \lambda\xi$, we have $f(x) = f(u) + \lambda f(\xi) = \lambda f(\xi)$ and $\langle x, \xi \rangle = \langle u, \xi \rangle + \lambda \langle \xi, \xi \rangle = \lambda$. Consequently,

$$f(x) = \lambda f(\xi) = \langle x, \xi \rangle f(\xi) = \langle x, \overline{f(\xi)}\,\xi \rangle.$$

Hence $f = \langle \cdot, v \rangle$ where $v := \overline{f(\xi)} \cdot \xi$.

Remark. The Riesz representation theorem states that a Hilbert space is self-dual. This is exploited in Dirac's bra und ket notation in quantum mechanics.

Completion of a metric space (1)

If a metric space (X, d) is not complete, it can be completed by adding more points to it (namely the missing limits of nonconvergent Cauchy sequences in X). This can be done in the following way:

- Call two Cauchy sequences (x_k) and (y_k) in X equivalent if $d(x_k, y_k) \to 0$ as $k \to \infty$. Denote by $\widehat{X} := \{\text{Cauchy sequences in } X\}/\sim$ the set of resulting equivalence classes.
- Define a metric $\widehat{d}$ on $\widehat{X}$ by $\widehat{d}([x_k],[y_k]) := \lim_{k\to\infty} d(x_k, y_k)$. This limit exists because $|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n) \to 0$ so that $\big(d(x_n, y_n)\big)$ is a Cauchy sequence in $\mathbb{R}$, hence convergent. The limit is also independent of the particular choice of representatives because if $(x_n) \sim (\xi_n)$ and $(y_n) \sim (\eta_n)$ then $|d(\xi_n, \eta_n) - d(x_n, y_n)| \leq d(\xi_n, x_n) + d(\eta_n, y_n) \to 0$. Thus $\widehat{d}$ is a well-defined mapping. It is easy to check that $\widehat{d}$ is indeed a metric on $\widehat{X}$.
- Interpret (X, d) as a subspace of $(\widehat{X}, \widehat{d})$ by identifying $x \in X$ with $[(x, x, x, \ldots)]$.

Completion of a metric space (2)

- Then X is dense in $\widehat{X}$. In fact, let $\xi \in \widehat{X}$, say $\xi = [(x_1, x_2, x_3, \ldots)]$. Let $x^{(n)} = [(x_n, x_n, x_n, \ldots)] \in X$. Then $\widehat{d}(x^{(n)}, \xi) = \lim_k d(x_k^{(n)}, \xi_k) = \lim_k d(x_n, x_k) \to 0$ as $n \to \infty$ so that $x^{(n)} \to \xi$.
- The space $\widehat{X}$ thus constructed is complete. In fact, let $(\xi^{(n)})$ be a Cauchy sequence in $\widehat{X}$. Since X is dense in $\widehat{X}$, there is $x^{(n)} = [(x_n, x_n, x_n, \ldots)]$ in X such that $\widehat{d}(\xi^{(n)}, x^{(n)}) < 1/n$. Then $d(x_m, x_n) = \widehat{d}(x^{(m)}, x^{(n)}) \leq \widehat{d}(x^{(m)}, \xi^{(m)}) + \widehat{d}(\xi^{(m)}, \xi^{(n)}) + \widehat{d}(\xi^{(n)}, x^{(n)}) < (1/m) + \widehat{d}(\xi^{(m)}, \xi^{(n)}) + (1/n) \to 0$ as $m, n \to \infty$. Hence (x_n) is a Cauchy sequence in X. Let $\xi := [(x_1, x_2, x_3, \ldots)] \in \widehat{X}$. Then $\widehat{d}(\xi^{(n)}, \xi) \leq \widehat{d}(\xi^{(n)}, x^{(n)}) + \widehat{d}(x^{(n)}, \xi) < (1/n) + \widehat{d}(x^{(n)}, \xi) \to 0$ as $n \to \infty$.

We have shown that $\widehat{X}$ is a completion of X, that is, $(\widehat{X}, \widehat{d})$ is an extension of (X, d) and is complete, and X is dense in $\widehat{X}$.

Completion of a metric space (3)

Theorem. Let $f : X \to Y$ be a uniformly continuous mapping between metric spaces, and let $\widehat{X}$ and $\widehat{Y}$ be completions of X and Y. Then there is a unique continuous extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ of f, and $\widehat{f}$ is itself uniformly continuous.

Proof. Being a uniformly continuous function, f maps Cauchy sequences to Cauchy sequences. Hence we can (unambiguously) define $\widehat{f}(\lim_{n\to\infty} x_n) := \lim_{n\to\infty} f(x_n)$ (and indeed must do so to make $\widehat{f}$ continuous). A routine verification shows that $\widehat{f}$ is indeed uniformly continuous.

Applying the above theorem to the identity mapping $X \to X$ shows that any two completions of X are isometric (with a unique isometry extending the identity map of X). Hence the completion of a metric space is unique up to isometry.

Completion of normed spaces and inner product spaces

Theorem. Let $(\widehat{X}, \widehat{d})$ be a completion of (X, d). If X is a normed space then so is $\widehat{X}$. If X is an inner product space then so is $\widehat{X}$.

Proof. Let (X, d) be a normed space so that d is translation-invariant and homogeneous. Then

$$\widehat{d}(\widehat{x}+\widehat{a}, \widehat{y}+\widehat{a}) = \lim_{n\to\infty} d(x_n+a_n, y_n+a_n) = \lim_{n\to\infty} d(x_n, y_n) = \widehat{d}(\widehat{x}, \widehat{y})$$

and

$$\begin{aligned} \widehat{d}(\lambda\widehat{x}, \lambda\widehat{y}) &= \lim_{n\to\infty} d(\lambda x_n, \lambda y_n) = \lim_{n\to\infty} |\lambda|\, d(x_n, y_n) \\ &= |\lambda| \lim_{n\to\infty} d(x_n, y_n) = |\lambda|\, \widehat{d}(\widehat{x}, \widehat{y}). \end{aligned}$$

Hence $\widehat{d}$ is also translation-invariant and homogeneous, hence is derived from a norm, namely $\|\!|\widehat{x}|\!\| = \widehat{d}(\widehat{x}, 0) = \lim_n d(x_n, 0) = \lim_n \|x_n\|$. Thus $\widehat{X}$ is a normed space. If (X, d) is even an inner product space, then $\|\cdot\|$ satisfies the parallelogram law. But then so does $\|\!|\cdot|\!\|$, so that $\widehat{X}$ is also an inner product space.