I Hilbert spaces, closed subspaces and orthogonal projections

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I Hilbert spaces, closed subspaces and orthogonal projections

V normed linear space

$f : V \to V$ linear

Is every linear f necessarily continuous?

Example

$V = \mathbb{R}^{(\mathbb{N})} = \{ (a_n) \in \mathbb{R}^{\mathbb{N}} \mid a_n \neq 0 \text{ for only finitely many } n \}$

$\| a_n \| = \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}}$

$e_n = (0, \dots, 0, 1, 0, \dots)$ basis for V

$f : V \to V$
$e_n \mapsto n^2 e_n$

$x_n := \frac{1}{n} e_n$, $\|x_n - 0\| = \|x_n\| = \frac{1}{n} \to 0 \Rightarrow x_n \to 0$

but $\| f(x_n) \| = \| n^2 \cdot \frac{1}{n} e_n \| = n \to \infty$

f is <u>not</u> continuous!

<u>Definition</u>

$f: V \to V$ linear

f is called bounded $\Leftrightarrow$ $\exists\, K > 0$ such that

$$\|f(x)\| \leq K \cdot \|x\| \qquad \forall x \in V$$

$$\|f\| := \sup_{\|x = 1\|} \|f(x)\|$$ is the norm of the linear operator f

<u>Theorem</u>

$f: V \to V$ linear

f is continuous $\Leftrightarrow$ f is bounded (see lecture by Karlheinz Spindler)

<u>Proof</u>:

"$\Rightarrow$" f continuous $\Rightarrow$ f continuous ~~at~~ at 0

$\exists\, \delta > 0$ such that $\|x\| \leq \delta \to \|f(x)\| \leq 1$

Let $x \neq 0$.

$$\|f(x)\| = \left\| \|x\| f\left(\frac{x}{\|x\|}\right) \right\| = \|x\| \cdot \left\| f\left(\frac{x}{\|x\|}\right) \right\|$$

$$= \frac{\|x\|}{\delta} \underbrace{\left\| f\left(\delta \frac{x}{\|x\|}\right) \right\|}_{\leq 1}$$

$$\leq \frac{1}{\delta} \|x\|$$

"$\Leftarrow$" f bounded

$x_n \to x$

$$\| f(x_n) - f(x) \| = \| f(x_n - x) \|$$

$$\leq \|f\| \cdot \|x_n - x\| \to 0 \quad \checkmark$$

Remark V finite-dim. Then every linear $f: V \to V$ is bounded.

Remark

V finite-dim. vector space with scalar product $\langle \cdot, \cdot \rangle$

$W \subseteq V$ subspace

V can be decomposed into

$$V = W \oplus W^\perp$$

$x \in V$

$$x = x_W + x_\perp$$

$$P: V \to W, \quad x \mapsto x_W$$ orthog. projection onto W

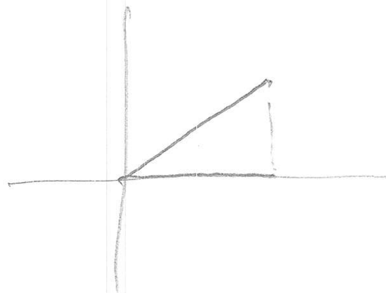
$$1 - P: V \to W^\perp, \quad x \mapsto x_\perp$$ orthog. projection onto $W^\perp$

Example

$V = \mathbb{R}^3$

$f: V \to V$

$(x_1, x_2, x_3) \mapsto (x_1, 0, 0)$ orth. projection onto the first coordinate



What about orth. projections in infinite-dim. vector spaces?

Example

$V = \ell_2 = \{ (a_n) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} a_n^2 < \infty \}$

$\langle a_n, b_n \rangle = \sum_{n=1}^{\infty} a_n \cdot b_n$

Let $W = \{ (a_n) \in \mathbb{R}^{\mathbb{N}} \mid a_n \neq 0$ for only finitely many $n \}$

W is a linear subspace of ℓ_2.

Take any $x = (a_n) \in \ell_2$.

Define $\tilde{a}_n := (a_1, \dots, a_n, 0, \dots)$.

$(\tilde{a}_n) \in W \qquad (\tilde{a}_n) \to x$

W is dense in l_2.

Is there an orthogonal projection $P: V \to W$?

Take $x \in V$.

Assume there is a decomposition $\underset{\in W}{x = x_W} + \underset{\in W^\perp}{x_\perp}$

$$\|x\|^2 = \|x_W\|^2 + \|x_\perp\|^2 \quad \Rightarrow \quad \|x_W\| \leq \|x\|$$

$$\|Px\| = \|x_W\| \leq \|x\|$$

Such P would be a bounded operator.

$$\mathbb{1}: V \to V \quad \text{bounded}$$
$$x \mapsto x$$

$$\mathbb{1} - P : V \to W^\perp$$
$$x \mapsto x^\perp \quad \text{bounded}$$

Consider $(\tilde{a}_n) \in W$.

$$(\tilde{a}_n) \to (a_n)$$

$$(\mathbb{1} - P)(\tilde{a}_n) = 0 \quad \forall n \in \mathbb{N}$$

$$(\mathbb{1} - P) \underset{\notin W}{x} \neq 0 \quad \text{if } x \in V \setminus W$$

$$(\tilde{a}_n) \to x$$

but $(\mathbb{1} - P)(\tilde{a}_n) \not\to (\mathbb{1} - P)x$ $\lightning$ W does not have an orth. complement $W^\perp$

Theorem V Hilbert space, W closed subspace.

Then W has an orth. complement $W^\perp$, and $V = W \oplus W^\perp$.

Proof: see lecture by Karlheinz Spindler

II Conditional expectations

Example

tossing of a coin, infinitely often

H = heads $\quad T$ = tails $\quad P(H) = P(T) = \frac{1}{2}$

$\Omega_0 = \{H, T\}$

$\Omega = \{ (\omega_k)_{k \in \mathbb{N}} \mid \omega_k \in \Omega_0 \}$

$\mathcal{A} = \mathcal{P}(\Omega)$ σ-field

We want to describe what happens during the first n experiments

$\mathcal{A}_n = \{ A_1 \times \dots \times A_n \times \Omega_0 \times \Omega_0 \times \dots \mid A_j \subseteq \Omega_0 \text{ for } 1 \leq j \leq n \}$

sub σ-field describing the first n experiments

$\mathcal{A}_n \subseteq \mathcal{A}_m$ if $n \leq m$

players A and B

A receives 1 from B if "heads"

-1 from B if "tails"

gain / loss for A after n experiments

$$S_n = \sum_{k=1}^{n} \left(1_H(\omega_k) - 1_T(\omega_k) \right)$$

$$S := \lim_{n \to \infty} S_n$$

S is well-defined for each $\omega = (\omega_k)$, potentially $+\infty$ or $-\infty$

S_n is $\mathcal{A}_n$-measurable

Can we describe S in terms of S_n in a measure-theoretic sense?

<u>Remark</u>

$(\Omega, \mathcal{A}, P)$ probability space

$\mathcal{A}_0 \subseteq \mathcal{A}$ sub σ-field

consider $L^2(\Omega) = \{ X : \Omega \to \mathbb{R},\ X\ \mathcal{A}\text{-measurable} \mid \int X^2 dP < \infty \}$

$L_0^2(\Omega) = \{ X \in L^2(\Omega) \mid X\ \mathcal{A}_0\text{ measurable} \}$

$L_0^2(\Omega)$ is a linear subspace of $L^2(\Omega)$.

It is a closed subspace:

Let $(X_n) \in L_0^2$, $X_n \to X$ (in L^2-sense).

Then a suitable subsequence converges pointwise to X.

Since limits of $\mathcal{A}_0$-measurable functions are $\mathcal{A}_0$-measurable, X is $\mathcal{A}_0$-measurable.

Consider the orthogonal projection of a random variable $X \in L^2(\Omega)$ onto $L_0^2(\Omega)$, written as $X_0 = P(X)$.

$$X = X_0 + X_\perp$$

Now take $A \in \mathcal{A}_0$. $\mathbb{1}_A$ is $\in L_0^2(\Omega)$.

$$\langle X, \mathbb{1}_A \rangle = \langle X_0 + X_\perp, \mathbb{1}_A \rangle = \langle X_0, \mathbb{1}_A \rangle + \underbrace{\langle X_\perp, \mathbb{1}_A \rangle}_{=0}$$

$$\int_A X \, dP = \int_\Omega X \cdot \mathbb{1}_A \, dP = \langle X, \mathbb{1}_A \rangle = \langle X_0, \mathbb{1}_A \rangle = \int_A X_0 \, dP \quad \forall A \in \mathcal{A}_0$$

<u>Definition</u>

$(\Omega, \mathcal{A}, P)$ probability space

$\mathcal{A}_0 \subseteq \mathcal{A}_0$ sub σ-field

$X : \Omega \to \mathbb{R}$ L^2

The conditional expectation of X w.r.t. $\mathcal{A}_0$ is any $\mathcal{A}_0$-measurable random variable that satisfies

$$\int_A X \, dP = \int_A Y \, dP \qquad \forall A \in \mathcal{A}_0.$$

Y is uniquely defined in L^2-sense (that is, P almost everywhere). We write $Y = E(X \mid \mathcal{A}_0)$.

Example

$\Omega = \{1, 2, 3, \dots, 6\}$, $\quad \mathcal{A} = \mathcal{P}(\Omega)$

$$
\begin{aligned}
X : \Omega &\to \mathbb{R} \\
1 &\mapsto 1 \\
2 &\mapsto 2 \\
&\vdots \\
6 &\mapsto 6
\end{aligned}
$$

$\mathcal{A}_0 = \{\emptyset, \Omega, \{1,2,3\}, \{4,5,6\}\}$

The conditional expectation $E(X \mid \mathcal{A}_0)$ must be of the form

$$
\left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} \mapsto a \qquad \left.\begin{matrix} 4 \\ 5 \\ 6 \end{matrix}\right\} \mapsto b \qquad , \quad a, b \in \mathbb{R}.
$$

$$
\int_{\{1,2,3\}} X \, dP = \int_{\Omega} X \cdot 1_{\{1,2,3\}} \, dP = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} = 1
$$

$$
\int_{\{1,2,3\}} dP = a \cdot \frac{1}{2} \qquad \Rightarrow \quad a = 2
$$

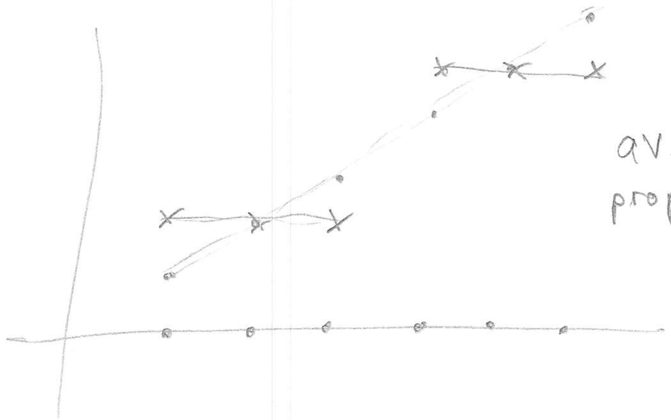
$$
\int_{\{4,5,6\}} X \, dP = (4+5+6) \cdot \frac{1}{6} = \frac{5}{2} \quad , \quad \int_{\{4,5,6\}} b \, dP = b \cdot \frac{1}{2}
$$

$$
\Rightarrow \quad b = 5
$$

$E(X \mid \mathcal{A}_0)$ is given by

$$\left.\begin{matrix}1\\2\\3\end{matrix}\right\} \mapsto 2$$

$$\left.\begin{matrix}4\\5\\6\end{matrix}\right\} \mapsto 5$$



<u>Proposition</u> (properties of taking conditional expectations)

$(\Omega, \mathcal{A}, P)$ probability space

$\mathcal{A}_0 \subseteq \mathcal{A}$ sub σ-field

$X, X_1, X_2 : \Omega \to \mathbb{R}$ random variables $\in L^2$

(a) $E(a_1 X_1 + a_2 X_2 \mid \mathcal{A}_0) = a_1 E(X_1 \mid \mathcal{A}_0) + a_2 E(X_2 \mid \mathcal{A}_0)$

(b) Let X be $\mathcal{A}_0$-measurable.

Then $E(X \mid \mathcal{A}_0) = X$

(c) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}$ sub σ-fields

Then $E(E(X \mid \mathcal{A}_0) \mid \mathcal{A}_1) = E(E(X \mid \mathcal{A}_1) \mid \mathcal{A}_0)$

$= E(X \mid \mathcal{A}_0)$

(d) $E(E(X \mid \mathcal{A}_0)) = E(X)$

(e) X independent of $\mathcal{A}_0$

Then $E(X \mid \mathcal{A}_0) = E(X)$

Proof: (a), (b), (c) obvious because of the projection property

(d) $$\int_A E(X \mid \mathcal{A}_0)\, dP = \int_A X\, dP \quad \forall A \in \mathcal{A}_0$$

Set $A = \Omega$

(e) Let $A \in \mathcal{A}_0$.

$$\int_A X\, dP = E(1_A \cdot X) = E(1_A) \cdot E(X)$$
$$= P(A) \cdot E(X)$$
$$= \int_A E(X)\, dP$$

Proposition (best approximation property)

$\mathcal{A}_0 \subseteq \mathcal{A}$ sub σ-field

Consider $L^2(\Omega)$ and $L^2_0(\Omega) = \{X \in L^2(\Omega) \mid X \text{ is } \mathcal{A}_0\text{-meas.}\}$

For each $X \in L^2(\Omega)$ there is uniquely one $Y \in L_0^0(\Omega)$ such that

$$\| X - Y \|_2 = \inf_{Z \in L_0^2} \| X - Z \|_2 .$$

This is $Y = E(X \mid \mathcal{A}_0)$.

Can we define conditional expectations $E(X \mid \mathcal{A}_0)$ for $X \in L^1$ as well?

<u>Proposition</u>

$(\Omega, \mathcal{A}, P)$ probability space

Then $L^2(\Omega) \overset{\text{dense}}{\subseteq} L^1(\Omega)$.

This also holds for any $(\Omega, \mathcal{A}, \mu)$ with $\mu(\Omega) < \infty$.

<u>Proof</u>

1st version ("by hand")

$$\int_\Omega |X| \, dP = \int_{\{|X| \geq 1\}} |X| \, dP + \int_{\{|X| < 1\}} |X| \, dP$$

$$\leq \int_{\{|X| \geq 1\}} |X|^2 \, dP + \int_{\{|X| < 1\}} |X| \, dP$$

$$\Rightarrow \|X\|_1 \leq \left(\|X\|_2\right)^2 + 1$$

$$\Rightarrow \left(X \in L^2 \Rightarrow X \in L^1 \right)$$

2^{nd} version (using Hölder's inequality)

$$\|X\|_1 = \|X \cdot 1\|_1 \leq \|X\|_2 \cdot \underbrace{\|1\|_2}_{=1}$$

We still have to prove "dense".

Let $X \in L^1$.

Define $X_n := X \cdot 1_{\{|X| \leq n\}}$, $X_n \in L^2$

Then $X_n \to X$ pointwise and in L^1 (by the dominated convergence theorem).

Thus L^2 is dense in L^1.

<u>Construction</u>

Let $X \in L^1$.

Define $X_n := X \cdot 1_{\{|X| \leq n\}} \in L^2$

$Y_n := E(X_n \mid \mathcal{A}_0)$ is $\in L^2$

Show that (Y_n) is a Cauchy sequence in L^1.

Let $Y := \lim Y_n$ and set $E(X \mid \mathcal{A}_0) := Y$.