

On the projectivity of proper normal curves over valuation domains

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Abstract

A theorem of S. Lichtenbaum states, that every proper, regular curve \mathcal{C} over a discrete valuation domain R is projective. This theorem is generalized to the case of an arbitrary valuation domain R using the following notion of regularity for non-noetherian rings introduced by J. Bertin: the local ring $\mathcal{O}_{\mathcal{C},x}$ of a point $x \in \mathcal{C}$ is called regular, if every finitely generated ideal I has finite projective dimension. The generalization is a particular case of a projectivity criterion for proper, normal R -curves: such a curve \mathcal{C} is projective if for every irreducible component Y of its closed fibre \mathcal{C}_m there exists a closed point P of the generic fibre of \mathcal{C} such that the Zariski closure \bar{P} meets Y , and meets \mathcal{C}_m in regular points only.

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1 Introduction and main results

In the present article integral, separated schemes \mathcal{C} of finite type over a valuation domain R are considered. Although these schemes are in general not noetherian, they share fundamental finiteness properties:

- the fibres $\mathcal{C}_p := \mathcal{C} \times_R k(p)$, $p \in \text{Spec}(R)$, are equidimensional of the same dimension d [11],
- \mathcal{C} is finitely presented over R [11],
- the structure sheaf $\mathcal{O}_{\mathcal{C}}$ is coherent [12].

If $d = 1$ such a scheme is said to be an R -curve.

Proper, normal R -curves appear in the article [5] as a means to understand the valuation theory of function fields of transcendence degree one. Moreover in [4] a connection between the projectivity of proper, normal R -curves and the

solvability of systems of algebraic diophantine equations is established. In the context of the present article the following result is remarkable:

(I) [4, Theorem 1]: *Let R be a valuation domain satisfying the Local Skolem Property. Then for every proper, normal R -curve \mathcal{C} with a geometrically integral generic fibre there exists a finite morphism $\mathcal{C} \rightarrow \mathbb{P}_R^1$; in particular \mathcal{C} is R -projective.*

For details about the Local Skolem Property the reader is referred to [4]. Every henselian valuation ring possesses this property.

The proof of (I) depends on certain valuation-theoretic results for function fields of transcendence degree one. Focusing on projectivity only this is somewhat in contrast to the situation in the noetherian case, where the singularities of \mathcal{C} play a prominent role as results by S. Lichtenbaum and J. Emsalem demonstrate:

(II) [10, Theorem 2.8]: *A proper, regular curve \mathcal{C} over a discrete valuation ring R is R -projective.*

(III) [2]: *Let \mathcal{C} be a proper curve over the discrete valuation ring R . If \mathcal{C} possesses only finitely many singular points contained in an open, affine set, then it is R -projective.*

Note that both results are actually and more generally valid for connected, flat R -curves. For valuation rings however being R -flat is equivalent to being torsion-free as an R -module. Therefore explicitly requiring flatness can be omitted, when considering integral R -schemes only.

The question arises, whether (II) and (III) can be generalized to the non-noetherian case, thus providing classes of projective R -curves other than the ones identified by (I). In the present article a generalization of Lichtenbaum's result (II) is given and concerning a generalization of (III) at least a first step is taken. To formulate the results precisely one has to use an appropriate notion of regularity for points of non-noetherian R -curves. For the convenience of the reader the relevant facts are summarized in the sequel.

Following J. Bertin [1] a local ring O is called regular, if for every finitely generated ideal $I \subseteq O$ the projective dimension $\text{pdim}(O/I)$ is finite. For a noetherian local ring this property is equivalent to the finiteness of the global homological dimension $\text{gldim}(O)$ and is thus a generalization of the noetherian notion of regularity. For a coherent local ring O it is known, that the finiteness of its weak homological dimension

$$\text{wdim}(O) := \sup(\text{fdim}(M) \mid M \text{ an } O\text{-module})$$

implies, but in general is not equivalent to regularity [3, Theorem 1.3.9]. Here the flat dimension $\text{fdim}(M)$ of an O -module M is defined to be the length $\ell \in \mathbb{N}$ of the shortest resolution

$$0 \rightarrow F_\ell \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

of M by flat O -modules F_i .

For an arbitrary scheme \mathcal{X} a point $x \in \mathcal{X}$ is said to be regular, if the local ring $\mathcal{O}_{\mathcal{X},x}$ is regular (in the sense of Bertin), otherwise the point is said to be singular. The regular locus $\text{Reg}(\mathcal{X})$ of \mathcal{X} is defined to be the set of regular points and the singular locus is $\text{Sing}(\mathcal{X}) := \mathcal{X} \setminus \text{Reg}(\mathcal{X})$.

In [8, Theorem 3.3] the regular locus of an R -curve \mathcal{C} is characterized as

$$\text{Reg}(\mathcal{C}) = \{x \in \mathcal{C} \mid \text{wdim}(\mathcal{O}_{\mathcal{C},x}) \leq 2\}. \quad (1)$$

The singular locus satisfies

$$\text{Sing}(\mathcal{C}) \subseteq \bigcup_{p \in \text{Spec}(R)} \text{Sing}(\mathcal{C}_p), \quad (2)$$

which follows from [3, Theorem 3.1.3]. Similar results are valid for R -schemes of relative dimension $d > 1$ – see [9].

Using these notions the main results of the present article can now be stated:

Theorem 1.1 *Let \mathcal{C} be a proper, normal curve over the valuation domain R satisfying the following condition: for every irreducible component Y of the closed fiber \mathcal{C}_m , $m \in \text{Spec}(R)$ the maximal ideal of R , there exists a closed point P on the generic fiber \mathcal{C}_0 , such that the Zariski-closure \overline{P} of P in \mathcal{C} has the properties*

$$\overline{P} \cap Y \neq \emptyset \text{ and } \overline{P} \cap \text{Sing}(\mathcal{C}) = \emptyset. \quad (3)$$

Then \mathcal{C} is R -projective.

The second part of condition (3) is equivalent with $\overline{P} \cap \text{Sing}(\mathcal{C}) \cap \mathcal{C}_m = \emptyset$: for $x \in \overline{P} \cap \text{Sing}(\mathcal{C})$ the inclusion $\overline{x} \subseteq \overline{P}$ holds and since the structure morphism $f : \mathcal{C} \rightarrow \text{Spec}(R)$ is closed, there exists $y \in \overline{x} \cap \mathcal{C}_m$. If $\mathcal{O}_{\mathcal{C},y}$ is regular, then $\mathcal{O}_{\mathcal{C},x}$ is regular too, because the latter ring is a localization of $\mathcal{O}_{\mathcal{C},y}$ and regularity is stable under localization.

With the help of Theorem 1.1 two classes of projective R -curves can be identified:

Theorem 1.2 *Let R be a valuation domain with maximal ideal m and let \mathcal{C} be a proper R -curve. Then R -projectivity of \mathcal{C} is implied by each of the following conditions:*

1. \mathcal{C} is regular,
2. R is henselian and $\text{Sing}(\mathcal{C}) \cap \mathcal{C}_m$ is finite.

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2 Divisorial ideals

In this section information about divisors on non-noetherian schemes relevant for the proof of Theorem 1.1 is presented in condensed form; the main source is the article [7].

Let \mathcal{X} be an integral scheme with coherent structure sheaf $\mathcal{O}_{\mathcal{X}}$ and let F be its field of rational functions. An $\mathcal{O}_{\mathcal{X}}$ -submodule $\mathcal{J} \subset \underline{F}$ of the constant sheaf \underline{F} over F is called fractional $\mathcal{O}_{\mathcal{X}}$ -ideal, if for every $x \in \mathcal{X}$ there exists an open neighborhood $U \subseteq \mathcal{X}$ of x and a section $s \in \mathcal{O}_{\mathcal{X}}(U) \setminus \{0\}$ such that $s \cdot \mathcal{J}|_U \subseteq \mathcal{O}_{\mathcal{X}}|_U$.

For fractional $\mathcal{O}_{\mathcal{X}}$ -ideals \mathcal{I} and \mathcal{J} the $\mathcal{O}_{\mathcal{X}}$ -modules $\mathcal{I} \cdot \mathcal{J}$ and $(\mathcal{I} : \mathcal{J})$ are fractional $\mathcal{O}_{\mathcal{X}}$ -ideals too. All $\mathcal{O}_{\mathcal{X}}$ -submodules $\mathcal{J} \subset \underline{F}$ of finite type are fractional $\mathcal{O}_{\mathcal{X}}$ -ideals and moreover they are coherent. Thus if \mathcal{I} and \mathcal{J} are both of finite type, then $\mathcal{I} \cdot \mathcal{J}$ and $(\mathcal{I} : \mathcal{J})$ are of finite type and hence coherent.

A fractional $\mathcal{O}_{\mathcal{X}}$ -ideal \mathcal{J} is called divisorial, if it satisfies

$$\mathcal{J} = \widehat{\mathcal{J}} := (\mathcal{O}_{\mathcal{X}} : (\mathcal{O}_{\mathcal{X}} : \mathcal{J})).$$

If \mathcal{J} is coherent, this property can be checked stalkwise, which implies that for a fractional $\mathcal{O}_{\mathcal{X}}$ -ideal \mathcal{J} the fractional $\mathcal{O}_{\mathcal{X}}$ -ideal $\widehat{\mathcal{J}}$ is divisorial. Consequently on the set $\mathcal{D}_{\text{coh}}(\mathcal{X})$ of coherent, divisorial $\mathcal{O}_{\mathcal{X}}$ -ideals one can define the product

$$\mathcal{I} \odot \mathcal{J} := \widehat{\mathcal{I} \cdot \mathcal{J}}, \tag{4}$$

giving $\mathcal{D}_{\text{coh}}(X)$ the structure of a commutative semigroup with neutral element $\mathcal{O}_{\mathcal{X}}$.

Every invertible $\mathcal{O}_{\mathcal{X}}$ -submodule $\mathcal{J} \subset \underline{F}$ is divisorial, since $\mathcal{J}^{-1} = (\mathcal{O}_{\mathcal{X}} : \mathcal{J})$. Invertible $\mathcal{O}_{\mathcal{X}}$ -submodules and Cartier divisors of \mathcal{X} can be identified and form a group $\text{CaDiv}(\mathcal{X})$ with respect to multiplication of fractional $\mathcal{O}_{\mathcal{X}}$ -ideals. For invertible $\mathcal{O}_{\mathcal{X}}$ -submodule multiplication and the product (4) coincide, so that $\text{CaDiv}(\mathcal{X})$ is a subgroup of $\mathcal{D}_{\text{coh}}(X)$.

With respect to the goal of the present article it is essential to have a criterion for the invertibility of a sheaf $\mathcal{J} \in \mathcal{D}_{\text{coh}}(X)$ at hand. Such a criterion can be derived from a generalization of the famous theorem of Auslander-Buchsbaum on the factoriality of (noetherian) regular local rings:

Theorem 2.1 *In a coherent, regular local ring finitely many elements always possess a greatest common divisor.*

A proof of this result can be found in [3, Corollary 6.2.10].

Corollary 2.2 *Let \mathcal{X} be an integral scheme with coherent structure sheaf. If $\mathcal{J} \in \mathcal{D}_{\text{coh}}(X)$ has the property*

$$\{x \in \mathcal{X} : \mathcal{J}_x \neq \mathcal{O}_{\mathcal{X},x}\} \subseteq \text{Reg}(\mathcal{X}),$$

then $\mathcal{J} \in \text{CaDiv}(\mathcal{X})$.

Proof. It suffices to show that every stalk \mathcal{J}_x as a $\mathcal{O}_{\mathcal{X},x}$ -module is generated by one element; to this end $\mathcal{J}_x \neq \mathcal{O}_{\mathcal{X},x}$ can be assumed. Due to the coherence of \mathcal{J} the equation

$$\mathcal{J}_x = (\mathcal{O}_{\mathcal{X}} : (\mathcal{O}_{\mathcal{X}} : \mathcal{J}))_x = (\mathcal{O}_{\mathcal{X},x} : (\mathcal{O}_{\mathcal{X},x} : \mathcal{J}_x))$$

holds. Therefore \mathcal{J}_x is a finitely generated, divisorial $\mathcal{O}_{\mathcal{X},x}$ -ideal. As such it satisfies

$$\mathcal{J}_x = \bigcap_{\mathcal{J}_x \subseteq f\mathcal{O}_{\mathcal{X},x}} f\mathcal{O}_{\mathcal{X},x}. \quad (5)$$

Since $s\mathcal{J}_x \subset \mathcal{O}_{\mathcal{X},x}$ for some $s \in F^*$, without loss of generality one can assume $\mathcal{J}_x \subset \mathcal{O}_{\mathcal{X},x}$. Let a_1, \dots, a_r be generators of \mathcal{J}_x . Then by Theorem 2.1 these elements possess a greatest common divisor b ; in other words $b\mathcal{O}_{\mathcal{X},x}$ is a principal overideal of \mathcal{J}_x , which is minimal with respect to inclusion. The representation (5) thus yields $\mathcal{J}_x = b\mathcal{O}_{\mathcal{X},x}$. \square

In the case of a normal R -curve $\mathcal{X} = \mathcal{C}$ divisorial $\mathcal{O}_{\mathcal{C}}$ -ideals can be obtained from closed points on the generic fiber \mathcal{C}_0 of \mathcal{C} :

Proposition 2.3 *Let \mathcal{C} be a normal curve over the valuation domain R . For every closed point P of the generic fiber \mathcal{C}_0 the following assertions hold:*

1. *The Zariski-closure \overline{P} of P in \mathcal{C} is a finitely presented, integral R -scheme with fibers of dimension zero.*
2. *The ideal sheaf $\mathcal{J}_{\overline{P}}$ inducing the reduced subscheme structure on \overline{P} is divisorial.*

This proposition follows from the more general result [7, Theorem 2.10]. There the normality of \mathcal{C} is used to show divisoriality of $\mathcal{J}_{\overline{P}}$. With an eye on Emsalem's result (III), that gets along without normality, it is remarkable that Proposition 2.3 is the only point in the present article, where the normality of the R -curves considered is used.

Point 1 of Proposition 2.3 implies, that the fibres \overline{P}_p , $p \in \text{Spec}(R)$, are finite, since \overline{P}_p is of finite type over $k(p)$. This fact is used in the following result needed in the proof of Theorem 1.2:

Proposition 2.4 *Let \mathcal{C} be a proper curve over the henselian valuation domain R . Then for every closed point P of the generic fiber \mathcal{C}_0 the intersection $\overline{P} \cap \mathcal{C}_m$ contains precisely one point.*

Proof. As a closed subscheme of an R -proper scheme \overline{P} is R -proper too. Hence applying [6, Théorème 8.11.1] yields, that \overline{P} is finite over R . Let A be the ring of global sections of \overline{P} , then $R \subseteq A \subset k(P)$ and the field extension $k(P)|K$, $K = \text{Frac}(R)$, is finite. Since the ring extension $A|R$ is finite, A is contained in the integral closure S of R in $k(P)$. Since R is henselian, S is a valuation domain, which implies that A is local. Hence \overline{P}_m consists of one point only. \square

3 Proofs of the main results

PROOF OF THEOREM 1.1: The proof is an adaptation of Lichtenbaum's proof for the projectivity of proper, regular curves over discrete valuation rings [10, section 2].

Let $f : \mathcal{C} \rightarrow \text{Spec}(R)$ be the structure morphism of a proper R -curve \mathcal{C} . In order to prove R -projectivity of \mathcal{C} one has to show the existence of an f -ample invertible sheaf \mathcal{L} on \mathcal{C} .

Let $g : \mathcal{C}_m \rightarrow \mathcal{C}$ be the natural morphism. Let \mathcal{L} be an invertible $\mathcal{O}_{\mathcal{C}}$ -module. Since the structure morphism f is of finite presentation, \mathcal{L} is f -ample if the sheaf $g^*(\mathcal{L})$ is ample on \mathcal{C}_m [6, Corollaire 9.6.4].

As Lichtenbaum points out in [10], $g^*(\mathcal{L})$ is ample if the stalks $g^*(\mathcal{L})_x$ are ideals in $\mathcal{O}_{\mathcal{C}_m, x}$ and for every irreducible component Y of \mathcal{C}_m , there exists a point $x \in Y$ such that $g^*(\mathcal{L})_x \neq \mathcal{O}_{\mathcal{C}_m, x}$.

Let Y_1, \dots, Y_r be the irreducible components of \mathcal{C}_m . By assumption there exist closed points P_1, \dots, P_r of the generic fiber \mathcal{C}_0 such that $\overline{P_k} \cap Y_k \neq \emptyset$ and $\overline{P_k} \cap \text{Sing}(\mathcal{C}) = \emptyset$. By point 2 of Proposition 2.3 and Corollary 2.2 the $\mathcal{O}_{\mathcal{C}}$ -ideal

$$\mathcal{L} := \mathcal{J}_{P_1} \cdot \dots \cdot \mathcal{J}_{P_r}$$

is invertible and the pullback $g^*(\mathcal{L})$ meets the requirements for being ample on the closed fiber \mathcal{C}_m . \square

PROOF OF THEOREM 1.2: The proof of both assertions is based on the following property of proper curves \mathcal{C} over a valuation domain R : given an irreducible component Y of the closed fiber \mathcal{C}_m , there exists a closed point $P \in \mathcal{C}_0$ such that $\overline{P} \cap Y \neq \emptyset$. In the noetherian case this property follows easily from the fact, that the local rings $\mathcal{O}_{\mathcal{C}, y}$ of closed points $y \in Y$ possess infinitely many prime ideals of height one. For a non-noetherian valuation domain R however there seems to be no short argument of this kind. Instead Zariski's Main Theorem is used to prove the existence of the point P .

Let $y \in Y$ be a closed point of Y . Let $U \subseteq \mathcal{C}$ be an affine, open neighborhood of y . The ring of sections $A := \mathcal{O}_{\mathcal{C}}(U)$ then is a finitely presented R -algebra.

Let $q \subset q_y$ be the prime ideals of A corresponding to the generic point of Y and to y respectively. The $k(m)$ -algebra A/q is finitely generated and has Krull dimension one, hence by Noether's Normalization Theorem there exists $\bar{t} \in A/q$ such that A/q is a finite extension of the polynomial ring $k(m)[\bar{t}]$.

Chose $t \in A$ with $t + q = \bar{t}$, then $R[t] \subseteq A$ is a polynomial ring. By construction $q \cap R[t] = m[t]$ and since A/q_y is a finite extension of $k(m)$, the prime ideal $p := q_y \cap R[t]$ is properly containing $m[t]$. Due to the finiteness of A/q over $k(m)[\bar{t}] = R[t]/m[t]$, the prime ideal q_y is minimal and maximal among the prime ideals of A lying over p . Zariski's Main Theorem thus yields the existence of a subring $B \subseteq A$ finite over $R[t]$ and an element $s \in B \setminus q_y$, such that $B_s = A_s$.

By the Lemma of Gauß there exists an irreducible polynomial $u \in p \setminus m[t]$; the prime ideal $uR[t]$ satisfies $uR[t] \cap R = 0$. By the Going-Down-Theorem in finite extensions, note that the polynomial ring $R[t]$ is integrally closed, there

exists a prime ideal $q_P \subset q_y \cap B$ lying over $uR[t]$. It follows that $q_P \mathcal{O}_{\mathcal{C},y}$ is a non-zero prime ideal lying over 0. The corresponding point $P \in \mathcal{C}_0$ then satisfies $y \in \overline{P}$.

Assertion 1: By [3, Lemma 6.2.6] and the subsequent remark, a proper, regular R -curve \mathcal{C} is a normal. Hence Theorem 1.1 is applicable: the condition (3) is satisfied as just shown.

Assertion 2: Let \mathcal{C} be a proper, normal R -curve. If $\mathcal{C}_m \cap \text{Sing}(\mathcal{C})$ is finite, one can choose a regular point y on every irreducible component of \mathcal{C}_m . As just proved there exists $P \in \mathcal{C}_0$ such that $y \in \overline{P} \cap \mathcal{C}_m$. If R is henselian Proposition 2.4 yields $\overline{P} \cap \mathcal{C}_m = \{y\}$ and therefore $\overline{P} \cap \text{Sing}(\mathcal{C}) = \emptyset$. Theorem 1.1 can now be applied. \square

References

- [1] J. Bertin, Anneaux cohérents réguliers, *C. R. Acad. Sc. Paris Ser. A* **273** (1971) 1–2.
- [2] J. Emsalem, *Projectivité des schémas en courbes sur un anneau de valuation discrète*, Bull. de la S.M.F. **101** (1973), 255–263.
- [3] S. Glaz, *Commutative coherent rings*, Lecture Notes in Math. **1371**, Berlin - Heidelberg - New York 1989.
- [4] B. Green, *On Curves over Valuation Rings and Morphisms to \mathbb{P}^1* , J. Number Theory **59** (1996), 262–290.
- [5] B. Green, M. Matignon, F. Pop, *On valued function fields III*, J. reine angew. Math. **432** (1992), 117–133.
- [6] A. Grothendieck, J. Dieudonné, *Eléments de géométrie algébrique IV*, Publ. Math. I.H.É.S **20**, **24**, **28**, **32** (1964, 1965, 1966, 1967).
- [7] H. Knaf, *Divisors on varieties over valuation domains*, Israel J. Math. **119** (2000), 349–377.
- [8] H. Knaf, *Regular Curves over Prüfer domains*, Fields Institute Communications **33** (2003), 89–106.
- [9] H. Knaf, *Regular local algebras over valuation domains: weak dimension and regular sequences*, J. Algebra Appl. **7** (5) (2008), 575–591.
- [10] S. Lichtenbaum, *Curves over discrete valuation rings*, Amer. J. Math. **90** (1968), 380–405.
- [11] M. Nagata, *Finitely generated rings over a valuation ring*, J. Math. Kyoto Univ. **5** (1965), 163–169.
- [12] G. Sabbagh, *Coherence of Polynomial Rings and Bounds in Polynomial Ideals*, J. Algebra **31**(1974), 499–507.